UNIVERSITY
OF SUSSEX

School of Mathematical & Physical Sciences
Department of Mathematics

Quantification of Mesoscopic and Macroscopic Fluctuations in Interacting Particle Systems

Panagiota Birmpa

Supervised by Dimitrios Tsagkarogiannis

Thesis submitted for the Degree of

Doctor of Philosophy in Mathematics

February 2018

Declaration

I hereby declare that this Thesis is submitted at the University of Sussex only, for the title of Doctor of Philosophy in Mathematics. I also declare that this Thesis was composed by myself, under the supervision of Dimitrios Tsagkarogiannis, and that the work contained therein is my own, except where stated otherwise, such as citations.

Brighton, May 21, 2018,

. . . .

(Panagiota Birmpa)

Abstract

The purpose of this PhD thesis is to study *mesoscopic* and *macroscopic* fluctuations in Interacting Particle Systems. The thesis is split into two main parts. In the first part, we consider a system of Ising spins interacting via Kac potential evolving with Glauber dynamics and study the macroscopic motion of an one-dimensional interface under forced displacement as the result of large scale fluctuations. In the second part, we consider a diffusive system modelled by a *Simple Symmetric Exclusion Process* (SSEP) which is driven out of equilibrium by the action of current reservoirs at the boundary and study the non-equilibrium fluctuations around the hydrodynamic limit for the SSEP with current reservoirs.

We give a brief summary of the first part. In recent years, there has been significant effort to derive deterministic models describing two-phase materials and their dynamical properties. In this context, we investigate the law that governs the power needed to force a motion of a one dimensional macroscopic interface between two different phases of a given ferromagnetic sample with a prescribed speed V at low temperature. We show that given the mesoscopic deterministic non-local evolution equation for the magnetisation (a non local version of the Allen-Cahn equation), we consider a stochastic *Ising spin system with Glauber dynamics and Kac interaction* (the underlying microscopic stochastic process) whose mesoscopic scaling limit (intermediate scale between microscale and macroscale) is the given PDE, and we calculate the corresponding large deviations functional which would provide the action functional. We obtain that by deriving upper and lower bounds of the large deviation cost functional. Concepts from statistical mechanics such as contours, free energy, local equilibrium allow a better understanding of the structure of the cost functional. Then we characterise the limiting behaviour of the action functional under a parabolic rescaling, by proving that for small values of the ratio between the distance and the time, the interface moves with a constant speed, while for

larger values the occurrence of nucleations is the preferred way to make the transition. This led to a production of two published papers [12] and [14] with my supervisor D. Tsagkarogiannis and N. Dirr.

In the second part we study the non-equilibrium fluctuations of a system modelled by SSEP with current reservoirs around its hydrodynamic limit. In particular, we prove that, in the limit, the appropriately scaled fluctuation field is given by a *Generalised Ornstein-Uhlenbeck process*. For the characterisation of the limiting fluctuation field we implement the Holley-Stroock theory. This is not straightforward due to the boundary terms coming from the nature of the model. Hence, by following a martingale approach (martingale decomposition) and the derivation of the equation of the variance for this model combined with "good" enough correlation estimates (the so-called v-estimates), we reduce the problem to a form whose Holley-Stroock result in [45] is now applicable. This is work in progress jointly with my supervisor and P. Gonçalves, [13].

Acknowledgements

First and foremost, I would like to thank my supervisor for the great support during my PhD, for the continuous guidance and the inspired motivation throughout the whole research. Besides my supervisor, I would like to thank N. Dirr and P. Gonçalves. The chance to collaborate with them was of great importance. I am also thankful to N. Georgiou for the nice discussions and his support during my PhD. Finally, I would like to thank my parents, Dimitris Birmpas and Sophia Pappa, my brother Giorgos, as well as Periklis for supporting me profoundly.

Contents

Chapter 1

Introduction

Statistical mechanics is concerned with the relation between prominent characteristics of large systems of interacting particles and properties of their microscopic constituents. Such characteristics include physical quantities such as the density of particles, the magnetisation, the interface between two phases etc. Mathematically, the evolution of these quantities in time can be described in different ways and with various scales: as a stochastic process in the atomistic scale (*microscopic*), a PDE (law of large numbers - hydrodynamic limit) at the *mesoscopic* level and e.g. a geometric evolution of the interface between the two phases at the *macroscopic* limit. Such systems can be either *isolated* (they do not exchange matter or energy with the outside world macroscopically) or *in contact to the outside environment*. Furthermore, by an appropriate scaling, it is also interesting to investigate the equilibrium or non-equilibrium, the stationary or dynamical fluctuations around the hydrodynamic limit (central limit theorem for the density of particles) and to characterise them.

The purpose of this PhD thesis is to study mesoscopic and macroscopic fluctuations in Interacting Particle Systems. The thesis is split into two main parts. In Part I, we consider a microscopic stochastic spin system and study the macroscopic motion of an one-dimensional interface under forced displacement, as the result of large scale fluctuations. In Part II, we consider a diffusive system, modelled by a *Simple Symmetric Exclusion Process* (SSEP), which is driven out of equilibrium by the action of *current reservoirs* at the boundary, and study the non-equilibrium fluctuations around its hydrodynamic limit.

Here, we give a brief summary of Part I. A large number of literature deals with the study of deriving deterministic model for two-phase materials and the system's dynamical properties. Our interest is to *investigate the most probable way a one dimensional macroscopic interface between two phases moves from an initial to a final position, within a fixed time*. Roughly, an interface between two phases is the boundary that separates the phases (*phase boundary*). In our model, the concept of interface is defined at the mesoscopic level, namely the stationary solutions of a given mesoscopic non local evolution equation (PDE), consisting of two homogeneous stationary solutions that play the role of the two phases, and of a third one which is inhomogeneous. The latter is what we define as interface. Thanks to [27, 48], we can rigorously relate its macroscopic evolution to a mesoscopic PDE which in turn is related to a lattice model of Ising-spins, with Glauber dynamics by the following multi-scale procedure: First, a spatial scaling of the order of the (diverging) interaction range of the Kac-potential, is applied to obtain a deterministic limit on mesoscale, which follows a nonlocal evolution equation mentioned above. This equation is then rescaled diffusively, to obtain the macroscopic evolution law, in this case motion by mean curvature. For an appropriate choice of the parameters, both limits can be done simultaneously, to obtain a macroscopic (and deterministic) evolution law for the phase boundary, in this case motion by mean curvature.

As our interface is one dimensional, the macroscopic evolution law for the phase boundary is an interface at rest. Under the above description, the problem we address, is reduced to two main questions. The first asks for the probability of macroscopic interfaces evolving differently for the deterministic limit law: to wit, a stationary solution is forced to move. At this stage, we derive an action functional as a large deviations functional. In fact, we derive quantitative estimates for the upper and the lower bound of the action functional, that penalises all possible deviations and obtains explicit error terms, which are also valid in the macroscopic scale. This work is in [12].

The second asks for the most probable way of such a motion. Therefore, in order to find the best mechanism for the macroscopic motion of the interface, one has to study the minimisers of the large deviations functional. To a better understanding of the structure of the functional in the sense that we reduce it in a simpler and more easily treatable form, we borrow concepts from statistical mechanics, such as *contours, free energy, local equilibrium, etc*, while for calculating its minimisers, the connection to the underlying stochastic process is also important. We prove that for small values of the ratio between the distance and the time, the interface moves with a constant speed, while for larger values, the occurrence of nucleations is the preferred way to make the transition (see Chapter 3).

Part II is devoted to dynamical non-equilibrium fluctuations for SSEP driven by current reservoirs. The Simple Exclusion Process (SEP) describes a system of particles, which move on a lattice by jumping to the nearest neighbouring sites independently of the others, except from jumps onto already occupied sites which are suppressed. To be precise, a particle at a position x waits, independently of the others, for an exponential distributed random time with parameter one, and then it chooses the site $x-1$ or $x+1$ with probability $p(x, x+1)$ (resp. $p(x, x-1)$). If $x+1$ (resp. $x-1$) is vacant, then it moves to $x+1$ (resp. $x-1$), otherwise it remains at the site x. In case $p(x, x+1) = p(x, x-1)$, for x in lattice, we say that the process is also symmetric (SSEP).

In our model, we consider the one dimensional SSEP in the interval $[-N, N]$, N is a positive integer, where each particle (independently) tries to jump to one of the nearest neighbouring sites at the rate $N^2/2$. In addition, we consider currents, that are obtained by driving forces that act on the boundaries. The driving forces could be physically interpreted in terms of reservoirs. In the model, we implement the so called "current reservoirs", namely we fix the current so that we send in particles from the right, at a rate which according to Fick's law has to be inversely proportional to the size of the system, and take out particles from the left at the same rate. For this reason, the rate is equal to $Nj/2$, where $j > 0$ is the external parameter which rules the birth-death mechanisms in the right-left boundaries.

There is significant work in the literature, studying problems of mass transport in a diffusive system under the action of the so called *density reservoirs* that is, systems in an interval where the reservoirs add and subtract particles from the right and left respectively at a unit rate, but instead of keeping the current fixed, they fix given densities close to the right and left (see [5, 10, 11, 30, 39] and the references therein). This also leads to a production of current from the high to low densities, according to Fick's law. The above model introduced by the authors in [22].

In [22, 23, 24], the authors study the hydrodynamic limit of the evolution of the density field, they establish the propagation of chaos and they study the stationary density field in the limit of the model. The purpose of [13] is to investigate the behaviour of the non-equilibrium fluctuations for the model. In particular, we wish to show that the sequence of the density fluctuation fields is tight, and every limiting point at time t can be written as the sum of a Gaussian random variable and the initial condition. Furthermore, if we assume that the sequence at time 0 converges to a mean-zero Gaussian process, then the limit of the sequence is unique, and is given by the *Ornstein-Uhlenbeck Process* with certain boundary conditions.

The thesis is organised as follows: Chapter 2 is an introductory chapter in order to establish notation and fix ideas. In particular, in Sect. 2.1, 2.2, 2.3 we discuss the Ising model, the mean field, and the Kac potentials, while in Sect. 2.3.1 the thermodynamic limit of the latter model is analysed, the known Lebowitz-Penrose limit. In Sect. 2.4, we briefly present the idea behind the infinite volume Gibbs measures and how they are connected to the phenomenon of phase transitions. Sect. 2.5 focuses on Glauber dynamics and especially Sect. 2.5.3 is devoted to Glauber dynamics with Kac potentials. Sect. 2.6 concerns with the mesoscopic theory and especially with the L-P functional and the gradient flow dynamics for this functional. We proceed to Part I where in Chapter 3 and 4, we present our papers [12, 14] respectively. Namely, in Sect. 3.1, we set up the model, in Sect. 3.3 we present the problem and the main results while in Sect. 3.3 and Sect. 3.4, we prove the main Theorem 3.4. The structure of Chapter 4 goes as follows: in Sect. 4.1 we state the main Theorem 3.6, Sect. 4.2 is a section with preliminaries, analysing the concepts of contours and multi-instanton manifold as well as further results based on them. In Sect. 4.3, we prove Theorem 3.6, while in Sect. 4.4 we discuss the particle model, the total cost and the total displacement. The last Sect. 4.5 is a complementary section to Sect. 4.3, including proofs of statements used in the main core of the proof of Theorem 3.6.

Part II is devoted to the simple symmetric exclusion process driven by current reservoirs. In the introduction of Part II, we present the exclusion process, we analyse the concept of the hydrodynamic limit and results for different models. Then, fluctuations around the hydrodynamic limit are also presented, giving a general sketch of how the limiting fluctuation field can be characterised, pointing out the tightness and the well-developed Holley-Stroock theory with which one can prove that the limiting fluctuation is given by the Ornstein-Uhlenbeck process. In Sect. 6.3, we define the fluctuation field for our model, while in Sect. 6.3.2 we compute the variance kernel at the microscopic level. To pass to the limit as $\epsilon \to 0$, we need the correlation estimates, the so-called v-estimates, that are presented in Sect. 6.3.3. This lead to the continuous variance kernel and this is done in Sect. 6.3.4. Finally, in Sect. 6.4 we compute explicitly the martingales coming from the martingale decomposition, while in Sect. 6.4.1 we explain why the space of test functions that we propose is the most probable.

Chapter 2

Elements of Statistical Mechanics

Introduction

This is an introductory chapter in Statistical Mechanics of Part I, and it focuses on giving a taste of what statistical mechanics is concerned with, and at the same time preparing the ground for Chapters 3 and 4. In this chapter we present results dealing with the macroscopic behaviour on the basis of its microscopic structure, rather than proving them. Our approach to the subject is mostly based on [17], [15], [36], [38], [40] and [60] and underlies the original idea of Maxwell, Boltzmann, and Gibbs: any macroscopic observable of a system can be understood by its microscopic picture. According to thermodynamics, it is enough to study a few macroscopic quantities (for example temperature, pressure for an ideal gas or magnetisation in magnetic systems), and any other quantity is completely determined by them. Even though the microscopic picture is very complex, by considering the suitable law of large numbers, one can reach the macroscopic behaviour.

In this spirit, we mostly discuss simplified mathematical models such as Ising model, mean-field model and Kac potentials that are presented in Sect 2.1, Sect. 2.2, and Sect. 2.3 respectively, are indicative models for the equilibrium statistical mechanics, due to their conceptual simplicity and their wide applicability, and their analysis is really insightful for a deep understanding. In Sect 2.4, we roughly present the main idea of R. L. Dobrushin, O. Lanford and D. Ruelle for the study of Gibbs measures on infinite volume systems. In Sect. 2.5, we approach the concept of dynamics, as it is essential to understand the mechanics that make a system flip from one microstate to another. In particular, we discuss the Glauber dynamics where their special physically-motivated flip rates allow a system to jump between microstates. Also, important results are presented when Kac potentials are endowed with Glauber dynamics. Finally, Sect. 2.6 is devoted to mesoscopic theory, and especially to L-P functionals as well as non local L-P evolution.

2.1 Ising Model

The Ising model is one the most famous models in Statistical Mechanics, due to the simplicity and richness of its behaviour. It was introduced by Wilhelm Lenz in 1920, aiming for a better understanding of the so-called phase transition phenomenon. The one-dimensional Ising model was solved by Ising in 1925, proving the non-existence of phase transitions, while the two dimensional Ising model with the absence of an external field was solved by Lars Onsager who proved rigorously the existence of a singularity of the pressure in the thermodynamic limit.

The idea of the Ising model is that a physical system can be interpreted abstractly by lattice arrangements of particles. Each particle has a spin oriented either up or down. Hence, the main objects are Ising configurations. Formally, an Ising spin configuration on $\mathbb{Z}^d$ or, on a bounded region $\Lambda \subset \mathbb{Z}^d$, $d \geq 1$, is a collection

$$\sigma = \{\sigma(x) : x \in \mathbb{Z}^d\}$$

with $\sigma(x) \in \{-1, +1\}$(up/down), which is the spin at the site x. An Ising configuration on Λ, σ_Λ, is defined analogously. The Ising phase space is $\mathcal{X} = \{-1, +1\}^{\mathbb{Z}^d}$ [resp. $\mathcal{X}_\Lambda = \{-1, +1\}^\Lambda$].

Remark 2.1. A physical system consists of an extremely large number of degrees of freedom, but not infinite. Therefore, it is natural to consider a configuration on Λ, which should be very large, rather than on $\mathbb{Z}^d$. However, from a mathematical point of view, the study of large objects usually undergoes appropriate limiting procedures. Hence technically, it is reasonable to consider infinite volume spaces. In general Statistical Mechanics concerns with this fundamental issue and for this reason, it postulates that systems with extremely large number of degrees of freedom can be well approximated by infinite systems.

We assume that interactions between particles are of short-range, in particular we consider interactions between particles of nearest neighbour (n.n) sites of the lattice. The interaction energy could depend on the spins at n.n sites in the following way: Given a *coupling constant* $J(x,y) := J\mathbb{1}_{y=x\pm1}$ with $J > 0$, when two particles at neighbouring sites are aligned (same spin direction), the interaction energy is $-J\sigma(x)\sigma(y)$ with $y = x \pm 1$. This means that the interaction energy tries to align the spin of the particles. In this case, we say that the system could lead to *spontaneous magnetisation* (all or most spins having the same direction).

On top of that, we also assume a constant external magnetic field, its intensity denoted by h, that is applied on the system. Then, for each $x \in \mathbb{Z}^d$ [resp. $x \in \Lambda$], the total energy of a configuration should include the terms $h\sigma(x)$ for each spin located at the site x. When the orientation of the external field points up, then the spins favour pointing up, otherwise spins favour pointing down. Hence, in bounded domains $\Lambda \subset \mathbb{Z}^d$, the total energy of a configuration $\sigma_\Lambda \in \mathcal{X}_\Lambda$ consists at least of the interaction energy between particles of neighbouring sites and the field energy of each spin in Λ. The picture should be reasonably the same for infinite volume systems. However, looking at the total field energy

$$-\sum_{x\in\mathbb{Z}^d} h\sigma(x)$$

there is no sense in defining such an energy. A natural way to think of energies on infinite volumes is to look at it through finite volumes. To be precise, let us consider more general coupling constants, that we will eventually be restricting ourselves to. Namely, we consider $J(x,y)$ such that J are symmetric ($J(x,y) = J(y,x)$), translational invariant ($J(x+z, y+z) = J(x,y)$) and summable ($\sum_{x\neq0} J(x,0) < \infty$). Then for any bounded region Λ, a configuration on $\mathbb{Z}^d$ can be written as $\sigma = (\sigma_\Lambda, \sigma_{\Lambda^c})$. Thus to define the energy, we have to take into account the interactions between spins in finite volume Λ and the "outside world" Λ^c. Then the energy is of Hamiltonian type and it is given by

$$H_{\Lambda,h}(\sigma) = -\frac{1}{2}\sum_{x\in\Lambda}\sum_{\substack{y\neq x,\\ y\in\Lambda}} J(x,y)\sigma(x)\sigma(y) - h\sum_{x\in\Lambda}\sigma(x) - \sum_{x\in\Lambda}\sum_{y\in\Lambda^c} J(x,y)\sigma(x)\sigma(y) \tag{2.1}$$

We see that

$$H_{\Lambda,h}(\sigma) = H_{\Lambda,h}(\sigma_\Lambda; \sigma_{\Lambda^c}) \tag{2.2}$$

where

$$H_{\Lambda,h}(\sigma_\Lambda;\sigma_{\Lambda^c}) = -\frac{1}{2}\sum_{x\in\Lambda}\sum_{\substack{y\neq x,\\ y\in\Lambda}} J(x,y)\sigma_\Lambda(x)\sigma_\Lambda(y) - h\sum_{x\in\Lambda}\sigma_\Lambda(x) - \\ -\sum_{x\in\Lambda}\sum_{y\in\Lambda^c} J(x,y)\sigma_\Lambda(x)\sigma_{\Lambda^c}(y) \quad (2.3)$$

This allows us to consider boundary conditions. In the case of (n.n), we have to see how particles in Λ interact with its boundary $\partial\Lambda$. The boundary conditions that are mostly used are the free, periodic boundary conditions, and configurations as boundary conditions. Depending on the boundary conditions, the energy has different expressions. For free boundary conditions, (no interaction between spins in and out of Λ),

$$H^{\text{free}}_{\Lambda,h}(\sigma) = -J\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda,\\ y=x\pm1}} \sigma(x)\sigma(y) - h\sum_{x\in\Lambda}\sigma(x) \quad (2.4)$$

For periodic boundary conditions, we consider the Ising model on torus $\mathbb{T}^d_N \simeq \otimes_d \mathbb{Z}/N\mathbb{Z}$. Then, the energy is given by

$$H^{\text{per}}_{\Lambda,h}(\sigma) = -J\sum_{\substack{x,y\in V_N,\\ \text{s.t}\,(x,y)\in\mathcal{X}_{V_N}}} \sigma(x)\sigma(y) - h\sum_{x\in V_N}\sigma(x) \quad (2.5)$$

where its set of vertices is denoted $V_N := \{0,\dots,N-1\}^d$ and the set of edges, $\mathcal{X}_{V_N} := \{(x,y) : x,y \in V_N \text{ and } \sum_{i=1}^d |(x_i - y_i)\ (\text{mod N})| = 1\}$. For configurations as boundary conditions, the situation is the following: For simplicity, let us consider the one dimensional Ising model. Then, given a configuration $\bar\sigma \in \mathcal{X}$, we consider the space $\mathcal{X}^{\bar\sigma}_\Lambda = \{\sigma \in \mathcal{X} : \sigma(x) = \bar\sigma(x),\ x \in \Lambda^c\}$, then the energy has the form

$$H^{\bar\sigma}_{\Lambda,h}(\sigma) \equiv H_{\Lambda,h}(\sigma;\bar\sigma) = -J\sum_{x\in\Lambda}\sum_{y=x\pm1}\sigma(x)\sigma(y) - h\sum_{x\in\Lambda}\sigma(x). \quad (2.6)$$

We note that some y's in the r.h.s of the first sum do not belong to Λ and therefore their values are specified by $\sigma(y) = \bar\sigma(y)$. For any type of boundary conditions mentioned above, a Gibbs distribution is assigned and it is given by

$$\mu^{(\text{bc})}_{\beta,E,h}(\sigma) = \frac{1}{Z^{(\text{bc})}_{\beta,E,h}} e^{-\beta H^{(\text{bc})}_{E,h}(\sigma)} \quad (2.7)$$

where

$$Z^{(\text{bc})}_{\beta,E,h} = \sum_{\sigma\in\mathcal{E}_{(bc)}} e^{-\beta H^{(\text{bc})}_{E,h}(\sigma)} \quad (2.8)$$

where β is the inverse temperature, (bc) could be free, $\bar{\sigma}$ or per, $E = \Lambda$ if (bc) is free or a configuration boundary condition and $E = V_N$ if (bc) is a periodic boundary condition. $\mathcal{E}_{(bc)}$ stands for a state space defined on the corresponding E. $Z^{(\mathrm{bc})}_{\beta,E,h}$ is a normalisation parameter which is known as *partition function*.

As mentioned, spins try to align locally (with the neighbours) and with the external magnetic field simultaneously. Since neighbouring spins alignment (locally) is favoured by the Ising hamiltonian, the coupling constant J is called ferromagnetic. The model described above is the classical Ising model.

To have a good understanding of the thermodynamics as well as phase transition of the Ising model, one has to exploit the partial information coming from the subregions Λ as discussed so far. This should happen by a limiting procedure as mentioned in Remark 2.1, approaches the infinite volume by sequences of growing finite subsets. This procedure is called *Thermodynamic limit*.

For simplicity, the notation (bc) refers to any of the above boundary conditions. To derive the thermodynamics, the crucial quantity is the partition function. To be specific, the *pressure* in a finite region $\Lambda \subset \mathbb{Z}^d$ is given by

$$p^{(\mathrm{bc})}_{\Lambda}(\beta, h) := \frac{1}{\beta|\Lambda|} \log Z^{(\mathrm{bc})}_{\beta,E,h} \tag{2.9}$$

It is easy to see that the function $(\beta, h) \mapsto p^{(\mathrm{bc})}_{\Lambda}(\beta, h)$ is convex for any type of boundary conditions (bc) and in addition $p^{(\mathrm{bc})}_{\Lambda}(\beta, h) = p^{(\mathrm{bc})}_{\Lambda}(\beta, -h)$ for (bc)=free or periodic, while for a fixed configuration $\bar{\sigma}$ as boundary conditions, $p^{\bar{\sigma}}_{\Lambda}(\beta, h) = p^{-\bar{\sigma}}_{\Lambda}(\beta, -h)$ holds.

Theorem 2.2. *In the thermodynamic limit, the pressure*

$$p(\beta, h) = \lim_{\Lambda \nearrow \mathbb{Z}^d} p^{(\mathrm{bc})}_{\Lambda}(\beta, h) \tag{2.10}$$

is well defined, independent of the sequence Λ and the boundary conditions (bc). It is also convex w.r.t β and h and an even function w.r.t the external field h.

In the theorem, by using the notation $\lim_{\Lambda \nearrow \mathbb{Z}^d}$, we mean that an increasing sequence of bounded regions of $\mathbb{Z}^d$, $\{\Lambda_n\}_{n\in\mathbb{N}}$, such that $\cup_{n\in\mathbb{N}}\Lambda_n = \mathbb{Z}^d$, have the following property:

$$\lim_{n\to\infty} \frac{|\{x \in \Lambda_n : \exists y \notin \Lambda \text{ s.t } x, y \text{ are (n.n)}\}|}{|\Lambda_n|} = 0$$

Then we say that $\{\Lambda_n\}_{n\in\mathbb{N}}$ converges to $\mathbb{Z}^d$ in *Van Hove sense*. For further details and proof, see [36]. Let us look at the magnetisation in $\Lambda \subset \mathbb{Z}^d$. We define the random variable,

$$m_\Lambda(\sigma) := \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma(x) \tag{2.11}$$

which is the *empirical magnetisation* or *magnetisation density* in Λ. We also define

$$m_\Lambda^{(\mathrm{bc})}(\beta,h) := \mu_{\beta,E,h}^{(\mathrm{bc})}(m_\Lambda) \tag{2.12}$$

where by $\mu_{\beta,E,h}^{(\mathrm{bc})}(m_\Lambda)$ we mean the expectation of $m_\Lambda(\sigma)$ w.r.t $\mu_{\beta,E,h}^{(\mathrm{bc})}$, hence $\mu_{\beta,E,h}^{(\mathrm{bc})}(m_\Lambda) = \sum_{\sigma\in\mathcal{E}_{(bc)}} m_\Lambda(\sigma)\mu_{\beta,E,h}^{(\mathrm{bc})}(\sigma)$. By differentiating the pressure in Λ w.r.t. h, we get

$$m_\Lambda^{(\mathrm{bc})}(\beta,h) = \frac{\partial p_\Lambda^{(\mathrm{bc})}(\beta,h)}{\partial h} \tag{2.13}$$

To understand the behaviour of $m_\Lambda^{(\mathrm{bc})}(\beta,h)$ in the limit, we have to be careful as there is a delicate point. First of all, it would be natural to define the average magnetisation density as $m(\beta,h) := \lim_{\Lambda\nearrow\mathbb{Z}^d} m_\Lambda^{(\mathrm{bc})}(\beta,h)$ and (2.13) would be inherited in the limit somehow. However, because of the connection magnetisation-pressure (2.13), the average magnetisation density may not be well-defined at every h, unless we secure that the pressure $p(\beta,h)$ is differentiable at every h. Indeed, due to the convexity of the pressure $p(\beta,h)$ w.r.t h, for every β the set

$$\mathcal{C}_\beta := \{h : p(\beta,h) \text{ is not differentiable w.r.t. h}\} \tag{2.14}$$

is at most countable. Regarding that, the correct definition of the average magnetisation density should be

$$m(\beta,h) := \lim_{\Lambda\nearrow\mathbb{Z}^d} m_\Lambda^{(\mathrm{bc})}(\beta,h), \text{ for every } h\notin\mathcal{C}_\beta. \tag{2.15}$$

It is well-defined, independent of the sequence Λ and the boundary conditions (bc) and its limiting behaviour is summarised in the next theorem:

Theorem 2.3. *For every $h\notin\mathcal{C}_\beta$, the average magnetisation density satisfies*

$$m(\beta,h) = \lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\partial p_\Lambda(\beta,h)}{\partial h} \tag{2.16}$$

and the function $h\mapsto m(\beta,h)$ is non-decreasing on $\mathcal{C}_\beta^c$, continuous at $h\notin\mathcal{C}_\beta$, and discontinuous at every $h\in\mathcal{C}_\beta$. The spontaneous magnetisation

$$m_\beta^* = \lim_{h\to0} m(\beta,h) \tag{2.17}$$

is always well-defined.

Theorem (2.3) gives rise to a first discussion about the phenomenon of phase transition. As we have already mentioned, for every β the pressure $p(\beta, h)$ could be non-differentiable at most of a countable number of points h. This means that for every β the average magnetisation density $m(\beta, h)$ could be discontinuous at most countable number of points h. In the basis of Statistical Mechanics the model exhibits a first-order phase transition at (β, h), if $h \mapsto p(\beta, h)$ is not differentiable at that point. For example in the one-dimensional Ising model, one can explicitly compute the pressure $p(\beta, h)$ and see that it is differentiable everywhere. Hence the average magnetisation $m(\beta, h)$ is everywhere continuous. As $h \to 0$, $m(\beta, h) \to 0$ and therefore, the spontaneous magnetisation $m^*(\beta)$ is equal to 0 for every $\beta > 0$. As $\beta \to +\infty$, the pressure is non-differentiable, that is

$$\lim_{\beta \to +\infty} m(\beta, h) = \begin{cases} +1, & \text{for } h > 0 \\ 0, & \text{for } h = 0 \\ -1 & \text{for } h < 0 \end{cases} \tag{2.18}$$

2.2 Mean field model

It is also interesting to study the Ising model with different types of coupling constants. The *Mean field model* (or *Curie-Weiss model*) is an indicative example of phase transitions and as the temperature varies, we observe different behaviours. In this model, each spin interacts with the other spins in the same way. Therefore, the positions of the particles do not play any role in the model (lack of geometry). To be precise, we consider a bounded region $\Lambda \subset \mathbb{Z}^d$, and the coupling constants are given by $J(x, y) = \frac{1}{|\Lambda|} \mathbb{1}_{x,y \in \Lambda}$, then the hamiltonian is defined as

$$\begin{aligned} H^{\mathrm{mf}}_{\Lambda,h}(\sigma_\Lambda) &:= -\frac{1}{2|\Lambda|} \left(\sum_{x \in \Lambda} \sigma_\Lambda(x) \right)^2 - h \sum_{x \in \Lambda} \sigma_\Lambda(x) \\ &= -|\Lambda| \left(\frac{1}{2} m_\Lambda(\sigma_\Lambda)^2 + h m_\Lambda(\sigma_\Lambda) \right) \end{aligned}$$

As we see, the empirical magnetisation defined in (2.11) appears naturally in $H^{\mathrm{mf}}_{\Lambda,h}(\sigma_\Lambda)$. This should lead to consider a set with all possible values of the magnetisation, that is

$$\mathcal{M}_\Lambda := \{-1, -1 + \frac{2}{|\Lambda|}, \dots, 1 - \frac{2}{|\Lambda|}, 1\} \tag{2.19}$$

and to define the *canonical partition function* in the following way:

$$Z^{\mathrm{can}}_{\beta,h,\Lambda,m,\zeta} := \sum_{\sigma_\Lambda \in \mathcal{X}_\Lambda} \mathbb{1}_{|m_\Lambda(\sigma_\Lambda) - m| < \zeta} e^{\beta|\Lambda|(\frac{m_\Lambda(\sigma_\Lambda)^2}{2} + h m_\Lambda(\sigma_\Lambda))} \tag{2.20}$$

for some accuracy parameter $\zeta > 0$ and $m \in \mathcal{M}_\Lambda$. The reason that we introduced the parameter ζ is because m is $|\Lambda|$-dependent and hence $\log \sum_{\sigma_\Lambda \in \mathcal{X}_\Lambda} \mathbb{1}_{m_\Lambda(\sigma_\Lambda)=m}(\sigma_\Lambda)$ is equal to ∞ for $m \notin \mathcal{M}_\Lambda$. The *mean field free energy* is

$$F_{\beta,h,\Lambda,m,\zeta} := -\frac{1}{\beta|\Lambda|} \log Z^{\text{can}}_{\beta,h,\Lambda,m,\zeta}. \tag{2.21}$$

For $m \in \mathcal{M}_\Lambda$, we also define

$$Z^{\text{can}}_{\beta,h,\Lambda,m,0} := \sum_{\sigma_\Lambda \in \mathcal{X}_\Lambda} \mathbb{1}_{m_\Lambda(\sigma_\Lambda)=m} e^{\beta|\Lambda|(\frac{m^2}{2}+hm)} \tag{2.22}$$

then the *effective mean field free energy*, $H^{\text{eff}}_{\Lambda,h}(m)$, is defined as

$$\beta H^{\text{eff}}_{\Lambda,h}(m) := -\log Z^{\text{can}}_{\beta,h,\Lambda,m,0} \tag{2.23}$$

It is easy to compute that

$$F^{\text{can}}_{\beta,h,\Lambda,m,0} = \frac{1}{\beta}\left(\left\{-\frac{m^2}{2} - hm\right\} - \frac{1}{\beta}\frac{1}{|\Lambda|}\binom{|\Lambda|}{|\Lambda|\frac{1+m}{2}}\right).$$

We are interested in understanding the behaviour of $F_{\beta,h,\Lambda,m,\zeta}$ in the limit as $|\Lambda| \to \infty$. By using Stirling's formula we conclude that for $m \in (-1,1)$

$$\lim_{\zeta \to 0} \lim_{|\Lambda| \to \infty} F_{\beta,h,\Lambda,m,\zeta} = \tilde{\phi}_{\beta,h}(m) \tag{2.24}$$

where

$$\tilde{\phi}_{\beta,h}(m) := \left\{-\frac{m^2}{2} - hm\right\} - \frac{1}{\beta}\mathcal{S}(m) \tag{2.25}$$

with

$$\mathcal{S}(m) := -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2} \tag{2.26}$$

being the *entropy* (*Cramer's entropy function*). Some of the properties of $\mathcal{S}(m)$ are the following: it is symmetric, convex and it takes its unique minimum value, 0, at 0. Its derivative lies on $(-1,1)$ and it is given by $I'(m) = -\frac{1}{2}\text{arctanh}\, m$. Moreover, there is a constant C such that for every $\Delta \in [-1,1]$ with $|\Delta| < 0.1$,

$$\max_{m,n \in \Delta} |\mathcal{S}(m) - \mathcal{S}(n)| \leq C|\Delta||\log|\Delta||$$

The next theorem is about the thermodynamic behaviour of the mean field model with the absence of magnetic external field.

Theorem 2.4. (i) *For $\beta \leq 1$, $\tilde{\phi}_{\beta,0}(m)$ is a symmetric function of m and convex.*

(ii) ***For** $\beta > 1$, $\tilde{\phi}_{\beta,0}(m)$ **is a double well shape function of** m **with minima at** $\pm m_\beta$ **(see Figure 2.2), where** m_β **is the unique positive solution of the mean field equation of***

$$m_\beta = \tanh \beta m_\beta. \tag{2.27}$$

More precisely, when

(i') $\beta < 1$, $\tilde{\phi}''_{\beta,0}(m) > 0$ ***for every*** $m \in (-1, 1)$,

(ii') $\beta = 1$, $\tilde{\phi}''_{1,0}(m) \geq 0$ ***for every*** $m \in (-1, 1)$ ***and*** $\tilde{\phi}''_{1,0}(0) = 0$,

(iii') $\beta > 1$, $\tilde{\phi}''_{\beta,0}(m) < 0$ ***for every*** $m \in (-1, 1)$ ***such that*** $|m| < \sqrt{1 - 1/\beta}$ ***is the spinodal region, while*** $\tilde{\phi}''_{\beta,0}(m) > 0$ ***for every*** $m \in (-1, 1)$ ***such that*** $|m| > \sqrt{1 - 1/\beta}$. ***The regions*** $(-m_\beta, -\sqrt{1 - 1/\beta})$ ***and*** $(\sqrt{1 - 1/\beta}, m_\beta)$ ***are the metastable regions whereas*** $|m| > m_\beta$ ***are the pure phase regions.***

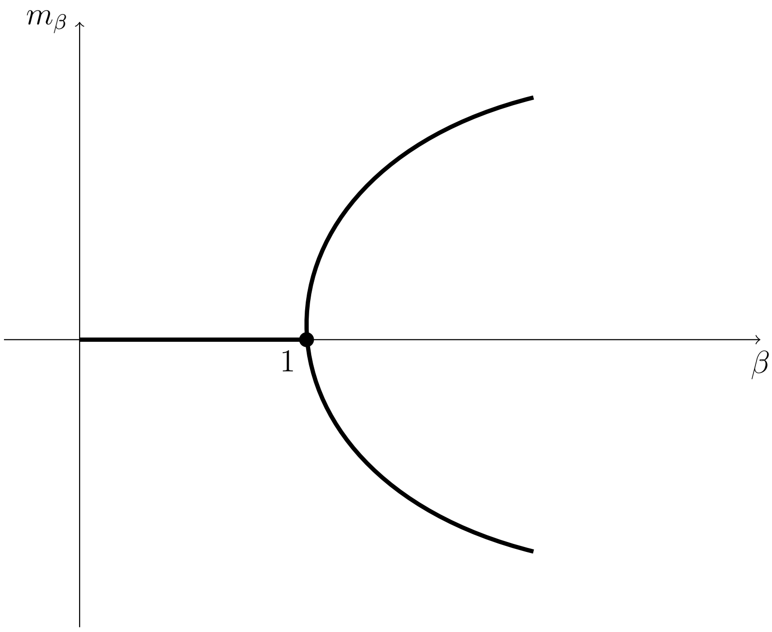

Figure 2.1: Phase Diagram of mean field model at $h = 0$ and critical temperature $\beta_c = 1$.

Before we discuss the mean field thermodynamics stated in Theorem 2.4, we consider the grand-canonical partition function

$$Z^{\text{grand}}_{\beta,h,\Lambda} := \sum_{m \in \mathcal{M}_\Lambda} \sum_{\sigma_\Lambda \in \mathcal{X}_\Lambda} \mathbb{1}_{m_\Lambda(\sigma_\Lambda) = m} e^{\beta|\Lambda|\left\{\frac{m^2}{2} + hm\right\}} \tag{2.28}$$

and then we define

$$P_{\beta,h,\Lambda} := \frac{1}{\beta|\Lambda|} \log Z^{\text{grand}}_{\beta,h,\Lambda}. \tag{2.29}$$

Then

Lemma 2.5.

$$\begin{aligned} \lim_{|\Lambda|\to\infty} P_{\beta,h,\Lambda} &= -\inf_{m\in[-1,1]} \left(\left\{ -\frac{m^2}{2} - hm \right\} - \frac{1}{\beta}\mathcal{S}(m) \right) \\ &:= g(\beta,h) \end{aligned} \tag{2.30}$$

We see that the Legendre transform of $\tilde{\phi}_{\beta,0}(m)$ is $g(\beta,h)$. Let us now define the probability distribution of random variable m_Λ

$$\mathbb{P}_{\beta,h,\Lambda}(m_\Lambda(\sigma_\Lambda) = m) = \frac{e^{\beta|\Lambda|\{\frac{m^2}{2}+hm\}}}{Z^{\text{can}}_{\beta,h,\Lambda}} \tag{2.31}$$

(it is inherited from the Gibbs distribution on the space of spin configurations). Then, the family $\{\mu_{\beta,h,\Lambda}\}_{\Lambda\subset\mathbb{Z}^d}$ satisfies the large deviations principle with rate function $\tilde{\phi}_{\beta,h}(m) + g(\beta,h)$. Given β and h, the equilibrium value of the magnetisation is the one that makes the rate function 0, that is $g(\beta,h) = -\tilde{\phi}_{\beta,h}(m)$ which is equivalent to $g(\beta,h) = -\tilde{\phi}_{\beta,0}(m)+hm$. This is the thermodynamic relation between the Gibbs and the Helmholtz free energy. Consequently we $g(\beta,h)$ is the *Gibbs free energy* while $\tilde{\phi}_{\beta,0}(m)$ is the *Helmholtz free energy*.

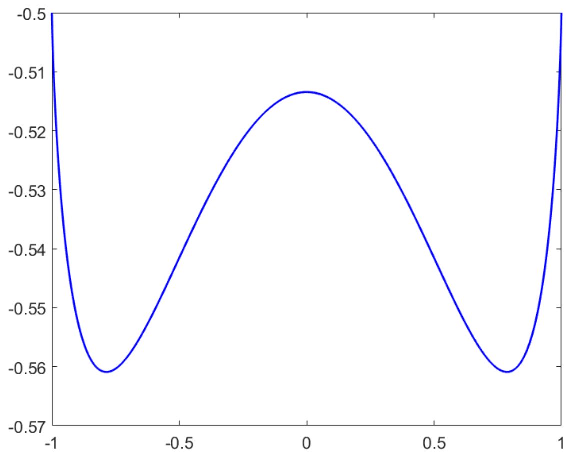

Figure 2.2: The graph of $\tilde{\phi}_{\beta,0}(m)$. The value at extremes is $-\frac{1}{2}$, while at the centre $-\frac{\log 2}{\beta}$.

Theorem 2.4 makes clear the competition between energy and entropy. For $\beta < 1$, entropy dominates and $\tilde{\phi}_{\beta,0}$ is strictly convex, while for $\beta > 1$, energy wins and $\tilde{\phi}_{\beta,0}$ is not convex any more. This is something that contradicts to thermodynamics, as it accepts the convexity of the free energy, loosing strictly convexity at the phase transition. More specifically, there is a flat region in the interval, with endpoints the magnetisation values of the coexisting pure phases. Here, the free energy has a double well shape with minima at $\pm m_\beta$. This comes from the fact that particles interact with each other with the same strength and the interaction range is of the same order as the size of the system. To correct this discrepancy, by using the Maxwell's equal area law, one replaces the non-convex free energy by its convex hull (maximal convex function not greater than $\tilde{\phi}_{\beta,0}$). However, it is worth to stress that from a probabilistic perspective, the probability distribution of m_Λ at any value between the pure phases of the system $\pm m_\beta$, is less probable than $\pm m_\beta$. Therefore, despite the fact that the interval $(-m_\beta, m_\beta)$ is the one that breaks down the convexity of the free energy when $\beta > 1$, all the values within are less probable to happen. This is summarised in the next theorem.

Theorem 2.6. *For $\beta > 1$, there exists $m_\beta > 0$ called spontaneous magnetisation, such that for all small enough $\epsilon > 0$, there exists $b = b(\beta, \epsilon) > 0$ such that for large enough Λ,*

$$\mu_{\beta,0,\Lambda}(m_\Lambda \in J_*(\epsilon)) \geq 1 - 2e^{-b|\Lambda|}$$

where $J_*(\epsilon) = (-m_\beta - \epsilon, -m_\beta + \epsilon) \cup (m_\beta - \epsilon, m_\beta + \epsilon)$. Hence, the mean field model provides ferromagnetism at low temperatures. For $\beta < 1$, for all small enough $\epsilon > 0$, there exists $c = c(\beta, \epsilon) > 0$ such that for large enough Λ,

$$\mu_{\beta,0,\Lambda}(m_\Lambda \in (-\epsilon, \epsilon)) \geq 1 - 2e^{-c|\Lambda|}$$

Consequently, at high temperatures the mean field model provides paramagnetism.

We have a similar analysis for $h \neq 0$. Namely, $-g(\beta, h) = \sup_{m \in [-1,1]} -\tilde{\phi}_{\beta,h}(m)$ and moreover $-g(\beta, h) = -\tilde{\phi}_{\beta,h}(m_\beta(h))$ that is, there is a value of magnetisation which depends on h, denoted by $m_\beta(h)$, where the supremum is attained. The value $m_\beta(h)$ has been computed by $\frac{\partial \tilde{\phi}_{\beta,h}}{\partial m} = 0$, equivalently it is the solution of the modified mean-field equation

$$m = \tanh \beta(m + h).$$

When $\beta < 1$, the solution is unique, while in case of $\beta > 1$ we could have more than one solution. More specifically if $h > 0$ then $m_\beta(h)$ is the largest solution. If $h < 0$, then

$m_\beta(h)$ is the smallest among its solutions. Furthermore, for $\beta < 1$

$$\lim_{h \to 0^+} m_\beta(h) = \lim_{h \to 0^-} m_\beta(h) = 0$$

whereas for $\beta > 1$,

$$\lim_{h \to 0^+} m_\beta(h) = m_\beta(h) > -m_\beta(h) = \lim_{h \to 0^-} m_\beta(h).$$

Consequently, the magnetisation has a jump discontinuity at $h = 0$ by $2m_\beta(h)$.

2.3 Kac Potentials

As we have already seen in the mean field model, the interactions between particles in a region $\Lambda \subset \mathbb{Z}^d$ are of type

$$J(x, y) = \frac{1}{|\Lambda|} \mathbb{1}_{x,y \in \Lambda}$$

Even though the model has rich phenomena (phase transitions, ferromagnetic behaviour at low temperatures and paramagnetic behaviour at high temperatures), the interactions are unphysical and responsible for the incongruity explained in Sect. 2.2, namely that the hamiltonian changes with volume. The same issue is present in the Van der Waals model in lattice gas language, where the Van der Waals assumption states that the system remains homogeneous (see [36], Chapter 4). That makes the theory unable to capture the inhomogeneities of the condensation. In fact, there are two hypotheses: first the particles interact repulsively at short distances and second the particles interact attractively at long distances. The first hypothesis formally restricts the particles to discrete regions so they do not share the same region. The attractive interaction of the second hypothesis is the one that contributes to the hamiltonian energy in an analogous way as in the mean field model. Maxwell's equal area rule again refines the Van der Waals assumption in such a way that inhomogeneities can be described.

Maxwell's construction can be employed when we are dealing with unnatural interactions and gives rigorously the right convexity of the free energy, as thermodynamics require. It is thus reasonable to seek for interactions that make the system behave as the mean field model (or Van der Waals model in gas lattice) in the limit, with the only difference that convexity of free energy will appear naturally in the limit as a result of the interaction itself. The first interesting part is how we could define such interactions, and the second one concerns with how well this type of interactions work to get the desired behaviour and properties.

We recall that in the mean field model, each spin interacts with all the other particles confined in region $\Lambda \subset \mathbb{Z}^d$ in the same way (homogeneity) and independently. We relax that by considering each spin to interact with other spins located in its neighbourhood, where this neighbourhood is of large diameter. This could be done by introducing a small parameter $\gamma > 0$, and neighbourhoods of spins has γ^{-1} diameter. Indeed, this is the idea of scaling. To be precise, the lattice spacing is 1, the range of interaction is γ^{-1} while the size of $\Lambda \subset \mathbb{Z}^d$ is $|\Lambda|$, and therefore we have to have $1 << \gamma^{-1} << |\Lambda|$. Hence the lattice spacing, the range of interactions and the size of the system are well separated, while in the mean field model we have a totally different picture where the range of the interactions is the same as the size of Λ. Moreover, as γ becomes smaller, more particles are included in a spin neighbourhood with γ^{-1} diameter and at the same time the strength of the interactions become weaker. That was the idea about refining the mean field assumption (respectively Van der Waals assumption) and was proposed by Kac with the known class of Kac potentials. For the purpose of the thesis, we restrict ourselves to them and especially to the following type of potentials that we are here after: By introducing the Kac scaling parameter, $\gamma > 0$, which is kept small, then

Definition 2.7. The coupling, $J_\gamma(x, y)$ is defined by

$$J_\gamma(x, y) = \gamma^d J(\gamma(x - y)), \quad x, y \in \Lambda, \tag{2.32}$$

where J is a function such that $J(r) = 0$ for all $|r| > 1$, $\int_{\mathbb{R}} J(r)dr = 1$ and $J \in C^2(\mathbb{R})$.

Hence, we have long range interactions (of order γ^{-1}), large connectivity of each site (of order γ^{-d}), small coupling constants (of order γ^d) and the total strength of a site is of order 1.

2.3.1 Lebowitz-Penrose limit

In this section we discuss the thermodynamic limit of the model. For the coupling constants given in (2.32), we consider the Hamiltonian

$$\begin{aligned} H_{\Lambda,h,\gamma}(\sigma_\Lambda; \bar{\sigma}_\Lambda) = -\frac{1}{2} \sum_{x,y\in\Lambda} J_\gamma(x, y)\,\sigma_\Lambda(x)\sigma_\Lambda(y) - h \sum_{x\in\Lambda} \sigma_\Lambda(x) \\ - \sum_{x\in\Lambda} \sum_{x\in\Lambda^c} J_\gamma(x, y)\sigma_\Lambda(x)\bar{\sigma}_{\Lambda^c}(y) \end{aligned} \tag{2.33}$$

where $\bar\sigma$ is a given configuration and

$$\mathcal{X}_\Lambda^{\bar\sigma} = \{\sigma \in \mathcal{X} : \sigma_{\Lambda^c}(x) = \bar\sigma_{\Lambda^c}(x),\ x \in \Lambda^{\rm c}\}$$

the state space with configuration $\bar\sigma$ as a boundary condition. The *grand-canonical pressure* in Λ is given by

$$p^{\bar\sigma}_{\beta,h,\gamma,\Lambda} = \frac{1}{\beta|\Lambda|}\log Z_{\beta,h,\gamma,\Lambda}(\bar\sigma_{\Lambda^c}), \quad Z_{\beta,h,\gamma,\Lambda}(\sigma_{\Lambda^c}) = \sum_{\sigma_\Lambda \in \mathcal{X}_\Lambda^{\bar\sigma}} e^{-\beta H_{\Lambda,h,\gamma}(\sigma_\Lambda;\bar\sigma_{\Lambda^c})}. \tag{2.34}$$

and the canonical free energy is

$$F^{\bar\sigma}_{\beta,\gamma,\Lambda}(m) = \frac{1}{\beta|\Lambda|}\log \tilde Z_{\beta,\gamma,\Lambda}(\bar\sigma_{\Lambda^c}), \quad \tilde Z^{\bar\sigma}_{\beta,\gamma,\Lambda}(\sigma_{\Lambda^c}) = \sum_{\substack{\sigma_\Lambda \in \mathcal{X}_\Lambda^{\bar\sigma},\\ m_\Lambda(\sigma)=m}} e^{-\beta H_{\Lambda,\gamma}(\sigma_\Lambda;\bar\sigma_{\Lambda^c})} \tag{2.35}$$

with $H_{\Lambda,\gamma}(\sigma_\Lambda;\bar\sigma_{\Lambda^c}) = H_{\Lambda,0,\gamma}(\sigma_\Lambda;\bar\sigma_{\Lambda^c})$.

Theorem 2.8. *For any $h \in \mathbb{R}$ and any $|m| < 1$,*

$$\lim_{\gamma\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^d} p^{\bar\sigma}_{\beta,h,\gamma,\Lambda} = g(\beta,h) \tag{2.36}$$

$$\lim_{\gamma\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^d} F^{\bar\sigma}_{\beta,\gamma,\Lambda}(m) = \sup_h\{hm - g(\beta,h)\} = CE\tilde\phi_{\beta,0} \tag{2.37}$$

where $g(\beta,h)$ is given in Lemma 2.30, $\tilde\phi_{\beta,0}$ is defined in (2.25) *and CE stands for the convex hull (we also call it as convex envelope).*

We refer for its proof to the literature, see [60]. In the preceding theorem we have a double limiting procedure. First, we look at the limiting behaviour of the grand-canonical pressure and the canonical free energy as $\Lambda \nearrow \mathbb{Z}^d$ in Van Hove sense and then as $\gamma \to 0$. The first limit for the pressure exists as we have seen in Theorem 2.2. General theory (see for example, [60], Chapter 4) also proves the existence of $\lim_{\Lambda\nearrow\mathbb{Z}^d} F^{\bar\sigma}_{\beta,\gamma,\Lambda}(m) = F^{\bar\sigma}_{\beta,\gamma}(m)$ as well. However, if we change the order in the limits, we are not led to interesting phenomena. Namely, It can readily be seen that

$$\lim_{\gamma\to 0} J_\gamma(x,y) = 0$$

and therefore

$$\lim_{\gamma\to 0} p^{\bar\sigma}_{\beta,h,\gamma,\Lambda} = -h + \frac{1}{\beta}\log(1+e^{2\beta h})^{|\Lambda|} := p_{\beta,h}$$

and

$$\lim_{\gamma\to 0} F^{\bar\sigma}_{\beta,\gamma,\Lambda}(m) = -\frac{1}{\beta}\binom{|\Lambda|}{|\Lambda|\frac{1+m}{2}} := F^{\bar\sigma}_{\beta,\Lambda}(m)$$

Then, $\lim_{\Lambda\nearrow\mathbb{Z}^d} F^{\bar{\sigma}}_{\beta,\Lambda}(m) = -\frac{1}{\beta}\mathcal{S}(m)$, where $\mathcal{S}(m)$ is defined in (2.26). Since $m = \frac{\partial p^{\bar{\sigma}}_{\beta,h,\gamma,\Lambda}}{\partial h}$, when $h > \frac{1}{2\beta},\ m(\beta,h) > 0$ [resp. $h \leq \frac{1}{2\beta},\ m(\beta,h) \leq 0$] and moreover $p^{\bar{\sigma}}_{\beta,h,\Lambda}$ is smooth for every $h \in \mathbb{R}$ which implies that m is continuous, which then implies that under the absence of attractive interactions, the model does not exhibit phase transitions. In addition, if we keep $\gamma > 0$ small and fixed, the range of the interactions, as well as the strenth, are kept finite. Then, we may loose phase transitions (see next section, Theorem 2.9).

Theorem 2.8 states that the Kac interactions in the limit, not only can describe the mean field model, but also give the right convexity property of the free energy naturally, as thermodynamics suggest. Kac potentials thus prove that the mean field assumption is not necessary, as the correct properties emerge physically from the interactions. This is a large deviations result, where the rate function of the total magnetisation of the model with Kac potentials converges to the convex hull of the corresponding rate function of mean field model in the limit of the infinite range interactions. The idea of the proof is of great significance, as concepts like spin blocks and coarse-grianing are implemented successfully. Loosely speaking, we look at the magnetisation of spins on cubes with appropriately chosen length (coarse-grained magnetisation). The Gibbsian probability of observing such magnetisations is given through a functional rate which is the free energy functional. Therefore, to calculate the probability, one has to minimise the functional (reduction to a variational problem). All this is presented in full detail in [60]. However, for the purpose of this thesis, we see fit to give the free energy functional: For $\gamma > 0$, we set

$$C_i^{(\gamma^{-1/2})} := \{r \in \mathbb{R}^d : i_k \leq r_k < i_k + \gamma^{-1/2}, k = 1, \ldots, d\}.$$

Then, we call

$$\mathcal{D}^{(\gamma^{-1/2})} := \{C_i^{(\gamma^{-1/2})} : i \in \gamma^{-1/2}\mathbb{Z}^d\}.$$

We say that a function $f(r)$ is $\mathcal{D}^{(\gamma^{-1/2})}$-measurable function if it is constant on each cube $C_i^{(\gamma^{-1/2})}$. We also say that a region Λ is $\mathcal{D}^{(\gamma^{-1/2})}$-measurable, if its characteristic is $\mathcal{D}^{(\gamma^{-1/2})}$-measurable function. Finally, we denote all $\mathcal{D}^{(\gamma^{-1/2})}$-measurable functions with values in $\{-1, -1 + \frac{1}{\gamma^{-1/2}}, \ldots, 1 - \frac{1}{\gamma^{-1/2}}, 1\}$ by $\mathcal{M}_\Lambda^{(\gamma^{-1/2})}$. Then, for $m_\Lambda \in \mathcal{M}_\Lambda^{(\gamma^{-1/2})}$ the free energy functional is given by:

$$F_{\gamma,\Lambda}(m_\Lambda|m_{\Lambda^c}) := F_{\gamma,\Lambda}(m_\Lambda) - \int_\Lambda \int_{\Lambda^c} J_\gamma(r,r')m_\Lambda(r)m_{\Lambda^c}(r')dr'dr \tag{2.38}$$

where

$$F_{\gamma,\Lambda}(m_\Lambda) := -\frac{1}{2}\int_\Lambda\int_\Lambda J_\gamma(r,r')m_\Lambda(r)m_\Lambda(r')dr'dr - h\int_\Lambda m_\Lambda(r)dr - \frac{1}{\beta}\int_\Lambda \mathcal{S}(m_\Lambda(r))dr. \tag{2.39}$$

In Sect. 2.6.1 we discuss further the above functional in a more general setting, by presenting several properties. The free energy functional given in (2.38) is known as *L-P functional* after Lebowitz and Penrose.

2.4 Infinite volume Gibbs measures

So far we have seen how inherent properties of extremely large systems can emerge by performing the thermodynamic limit. The question that we address in this section is how we can have a direct characterisation of the properties of a system in thermal equilibrium, without following a limiting procedure. To answer that, we focus on the characterisation of the equilibrium states of a system. As the equilibrium states can be described by Gibbs measures, the addressed question can be alternatively formulated as how we could define the Gibbs measures directly on a countably infinite lattice.

Let us start by considering the state space $\mathcal{X}_\Lambda$, when Λ is a very large region but finite (so large as the size of the actual size of the physical system), and then we eventually proceed to answering the main issue of the section. First of all, we need to define a probability measure on $\mathcal{X}_\Lambda$. The appropriate measure should be in agreement with the system in thermal equilibrium. Consequently, the first question under investigation is the following:

What is the proper choice of measure on $\mathcal{X}_\Lambda$ to describe a system in equilibrium?

A system in equilibrium is directly related to its Hamiltonian and once specified then the suitable choice of measure on $\mathcal{X}_\Lambda$ is the Gibbs measure and typically is of form

$$\mu_{\beta,h}(d\sigma) = \frac{1}{Z_{\beta,h}}e^{-\beta H_h(\sigma)}d\sigma \tag{2.40}$$

where β plays the role of inverse temperature, $Z_{\beta,h}$ a normalisation parameter and $d\sigma$ is an a priori measure on the phase space. We have seen that when the size of a lattice is finite, the hamiltonian is well-defined and (2.40) describes a system in equilibrium.

Going back to the initial question, when the size of the lattice is infinite, the way of defining the Hamiltonian should be different and therefore the Gibbs measure. In fact, by properly incorporating (2.3), we have a direct extension to the infinite lattices. The idea of R. L. Dobrushin, O. Lanford and D. Ruelle was to view the Hamiltonian as a function $H : (\Lambda, \sigma) \to H_\Lambda(\sigma)$ and define for any fixed configuration $\bar{\sigma} \in \mathcal{X}$, and for any finite $\Lambda \subset \mathbb{Z}^d$, the probability measure

$$\mu^{\bar{\sigma}}_{\beta,\Lambda,h}(d\sigma) = \frac{1}{Z_{\beta,\Lambda,h}(\bar{\sigma}_{\Lambda^c})} e^{-\beta H_{\Lambda,h}(\sigma_\Lambda;\bar{\sigma}_{\Lambda^c})} d\sigma \tag{2.41}$$

where the partition function is

$$Z_{\beta,\Lambda,h}(\bar{\sigma}_{\Lambda^c}) = \sum_{\sigma_\Lambda \in \mathcal{X}_\Lambda} e^{-\beta H_{\Lambda,h}(\sigma_\Lambda;\bar{\sigma}_{\Lambda^c})} \tag{2.42}$$

Then, a DLR measure, $\mu_{\beta,h}$ is a distribution of a stochastic process parametrised by the sites of a lattice, such that it admits prescribed versions of the conditional distributions with respect to the configurations outside finite regions (*DLR condition*), that is

$$\mu_{\beta,h}(d\sigma|\sigma_{\Lambda^c} = \bar{\sigma}_{\Lambda^c}) = \mu^{\bar{\sigma}}_{\beta,\Lambda,h}(d\sigma),$$

where $\mu^{\bar{\sigma}}_{\beta,\Lambda,h}$ is defined in (2.41). μ_β is called DLR measure thanks to R. L. Dobrushin, O. Lanford and D. Ruelle formalism.

However, all the above are pointless unless we secure the existence of such measures. For many systems existence is proven. The problem of uniqueness or non-uniqueness of DLR measures is also of paramount importance, as it is directly connected to phase transitions. The question of uniqueness can be addressed the following way: If $\mathcal{G}(\alpha)$ is the set of all DLR measures of a given family $\alpha = \{\mu^{\bar{\sigma}}_{\beta,\Lambda,h}\}_{\bar{\sigma},\Lambda}$, what conditions α has to have on in order to guarantee that $\mathcal{G}(\alpha)$ contains exactly one element. The fact that further conditions on the family α may ensure uniqueness lies on the fact that it contains all the information of dependencies between configurations on different parts of $\mathbb{Z}^d$ through the energies $H_{\Lambda,h}(\sigma_\Lambda;\bar{\sigma}_{\Lambda^c})$. This leads to Dobrushin's condition of weak dependence. Before we state the criterion, we give some definitions. We start by defining the *Vaserstein distance between the measures* $\mu^{\bar{\sigma}}_{\beta,\{x\},h}$ and $\mu^{\bar{\bar{\sigma}}}_{\beta,\{x\},h}$.

$$R(\mu^{\bar{\sigma}}_{\beta,\{x\},h}, \mu^{\bar{\bar{\sigma}}}_{\beta,\{x\},h}) = \inf_{Q_x} \sum_{s,s' \in \{-1,1\}} Q_x(s,s')|s-s'|$$

where Q_x is the coupling between the conditional probabilities $\mu^{\bar{\sigma}}_{\beta,\{x\},h}$ and $\mu^{\bar{\bar{\sigma}}}_{\beta,\{x\},h}$, that is

$$\sum_{s' \in \{-1,1\}} Q_x(s,s') = \mu^{\bar{\sigma}}_{\beta,\{x\},h}(\sigma(x) = s)$$

$$\sum_{s\in\{-1,1\}} Q_x(s,s') = \mu^{\bar{\bar{\sigma}}}_{\beta,\{x\},h}(\sigma(x)=s')$$

Theorem 2.9. [Dobrushin's condition of weak dependence] *Suppose that*

$$\sup_x \sum_{y\neq x} r(x,y) < 1, \quad \textit{with} \;\; r(x,y) := \frac{1}{2}\sup_\sigma R(\mu^{\bar{\sigma}}_{\beta,\{x\},h}, \mu^{\bar{\sigma}^y}_{\beta,\{x\},h}),$$

where $\bar{\sigma}^y(z) = -\bar{\sigma}(y)$ *if* $z = y$ *and* $\bar{\sigma}^{(y)}(z) = \bar{\sigma}(z)$ *if* $z \neq y$. *Then, the probability measure specified by* α *is unique:* $|\mathcal{G}(\alpha)| = 1$.

Thus, the size of the set $\mathcal{G}(\alpha)$ has a sensitive dependence on the nature of α. We can also conclude uniqueness if the following condition is satisfied:

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\neq x} \beta|J(x,y)| < 1$$

and the coupling constants $J(x,y)$ have the properties mentioned in Sect. 2.1. However, there are many situations where several infinite volume Gibbs measures exist for the same Hamiltonian and the same temperature. According to what we have said so far, this means that a system could have several distinct equilibrium states and we can say that this is equivalent to the existence of phase transitions. Analogously, uniqueness is equivalent to no phase transitions. We give some known results for specific models that exhibit or not phase transitions.

Theorem 2.10. *In* $d = 1$*, and for coupling constants* $J(x,y)$ *such that*

$$\sup_z \sum_{x<z} \sum_{z\geq y} |J(x,y)| < \infty$$

There is only one DLR measure for each $\beta > 0$.

The theorem covers also the case of Kac potentials with small but fixed $\gamma > 0$.

Theorem 2.11. *In* $d \geq 2$*, if* $h = 0$ *and* $\beta > 0$ *is large enough, then the n.n. ferromagnetic Ising model has a phase transition (at least two DLR measures).*

Theorem 2.12. *In any dimension and for any of the interactions considered in Sect. 2.1, there is a* $\beta_0 > 0$ *such that for any* $\beta < \beta_0$ *there is a unique DLR measure.*

Theorem 2.13. *In the Ising model with Kac potential with* $d \geq 2$*, given any* $\delta > 0$*, there is a* $\gamma(\delta) > 0$ *such that* $\beta_c(\gamma) < 1 + \delta$ *for any* $\gamma < \gamma(\delta)$*. In particular, for* $\beta > 1$*, and any* γ *small enough, there are two DLR measures.*

For their proofs we refer to [60], Chapter 3 and Chapter 9 and the references therein.

2.5 Glauber Dynamics

In this section we focus on the spin dynamics, and mostly on the dynamics on the one-dimensional Ising model with Kac interactions. As we have seen so far, all the equilibrium properties for the Ising model follow from the partition function. If one would like to drive the system out of equilibrium, the nature of spin dynamics plays a crucial role. There is no unique way to choose the right dynamics, as depending on the physical considerations of system, dynamics might be formulated differently.

A simple way to extend the Ising model out of equilibrium is the non-conservative single spin dynamics or the so called *Glauber dynamics*. Roughly speaking, by running Glauber dynamics, spins are selected one at a time in random order and each changes at a rate that depends on the change in the energy of the system as a result of this update. Due to the fact that only one spin flip occurs each time, the magnetisation in general is not conserved.

The structure of this section goes as follows: we define the concept of spin flip generator and the invariant measure. Then, in the context of the Ising model endowed with Glauber dynamics, we discuss the detailed balance property and reversibility. Then we proceed to the Glauber dynamics with Kac potentials where we present the deterministic behaviour of the system as $\gamma \to 0$.

2.5.1 Semigroups and generators for the spin flip dynamics

In this subsection, we discuss the spin flip semigroups for Ising models in bounded domains and for general Ising interactions, in order to prepare the ground for an extensive analysis on Kac interactions.

Definition 2.14. Let L be a linear operator on $L^\infty(\mathcal{X}_\Lambda)$ and $\{e^{Lt}\}_{t\geq 0}$ the semigroup generated by L. L is a *spin flip generator* if for every $f \in L^\infty(\mathcal{X}_\Lambda)$

$$Lf(\sigma_\Lambda) = \sum_{x\in\Lambda} c(x,\sigma_\Lambda)(f(\sigma_\Lambda^{(x)}) - f(\sigma_\Lambda)), \qquad c(x,\sigma_\Lambda) > 0 \tag{2.43}$$

where

$$\sigma^{(x)}(z) = \begin{cases} \sigma(z), & \text{if } z \neq x, \\ -\sigma(x), & \text{if } z = x. \end{cases} \tag{2.44}$$

$c(x, \sigma_\Lambda)$ is called *spin flip intensity at* x when the state is σ_Λ. Moreover, we denote $P_t(\sigma_\Lambda|\sigma^*_\Lambda) := e^{Lt}(\sigma^*_\Lambda, \sigma_\Lambda)$.

For every $\sigma^*_\Lambda \in \mathcal{X}_\Lambda$ and for every $t \geq 0$, $P_t(\cdot|\sigma^*_\Lambda)$ defines a probability on $\mathcal{X}_\Lambda$. It can be seen that

$$\frac{\partial}{\partial t} P_t(\sigma_\Lambda|\sigma^*_\Lambda) = \left[e^{Lt} L \mathbb{1}_{\sigma_\Lambda}\right](\sigma^*_\Lambda)$$

which implies that for fixed configuration σ^*_Λ the forward Kolmogorov equation is given by

$$\frac{\partial}{\partial t} P_t(\sigma_\Lambda|\sigma^*_\Lambda) = -\left(\sum_{x\in\Lambda} c(x, \sigma_\Lambda)\right) P_t(\sigma_\Lambda|\sigma^*_\Lambda) + \sum_{x\in\Lambda} c(x, \sigma^x_\Lambda) P_t(\sigma^x_\Lambda|\sigma^*_\Lambda)$$

The backward Kolmogorov equation is obtained by doing the opposite, that is we fix the configuration σ_Λ and we view the probability $P_t(\sigma_\Lambda|\sigma^*_\Lambda)$ as a function of σ^*_Λ. Next, we introduce the notion of the invariant or stationary measure.

Definition 2.15. Let ν be a measure on $\mathcal{X}_\Lambda$, then we say that ν *evolves under the semigroup* e^{Lt} if

$$\nu_t(f) = \nu(e^{Lt} f)$$

where $\nu(f)$ is the expectation of f w.r.t ν, that is

$$\nu(f) = \sum_{\sigma_\Lambda \in \mathcal{X}_\Lambda} \nu(\sigma_\Lambda) f(\sigma_\Lambda).$$

μ is *invariant under the semigroup* e^{Lt} if

$$\mu(f) = \mu(e^{Lt} f)$$

or equivalently for any $\sigma_\Lambda \in \mathcal{X}_\Lambda$

$$\mu(\sigma_\Lambda) = \sum_{\sigma^*_\Lambda \in \mathcal{X}_\Lambda} \mu(\sigma^*_\Lambda) P_t(\sigma_\Lambda|\sigma_{\Lambda^*}).$$

Theorem 2.16. *The following are equivalent:*

(i) *μ is invariant*

(ii) $\mu(Lf) = 0$ *for all* $f \in L^\infty(\mathcal{X}_\Lambda)$

(iii)

$$\sum_{x\in\Lambda} \mu(\sigma^{(x)}_\Lambda) c(x, \sigma^{(x)}_\Lambda) = \left(\sum_{x\in\Lambda} c(x, \sigma_\Lambda)\right) \mu(\sigma_\Lambda), \textit{ for all } \sigma_\Lambda.$$

Uniqueness of the invariant measure is guaranteed when the Döblin condition holds, that is for every $t > 0$

$$\inf_{\sigma_\Lambda, \sigma^*_\Lambda} P_t(\sigma_\Lambda | \sigma^*_\Lambda) > 0$$

which is valid if all the spin flip intensities are positive.

2.5.2 Glauber spin flip rates

Let Λ be a bounded region and σ_{Λ^c} a fixed boundary condition.

Definition 2.17. The Glauber dynamics is the spin flip process with generator (2.43) with the below spin flip intensities

$$c(x, \sigma_\Lambda) = c_0(x, \sigma_{\Lambda \setminus x}) e^{-\frac{\beta}{2} \Delta_x H(\sigma_\Lambda; \sigma_{\Lambda^c})}, \tag{2.45}$$

where

$$\Delta_x H(\sigma_\Lambda; \sigma_{\Lambda^c}) = H(\sigma^x_\Lambda; \sigma_{\Lambda^c}) - H(\sigma_\Lambda; \sigma_{\Lambda^c})$$

and $c_0(x, \sigma_{\Lambda \setminus x})$ may be any strictly positive function of $\sigma_{\Lambda \setminus x}$ and we call it mobility coefficient.

Theorem 2.18. *The Gibbs measure $\mu(\sigma_\Lambda) \equiv \mu^{\bar{\sigma}}_{\beta,\Lambda,h}(\sigma_\Lambda)$ at inverse temperature $\beta > 0$ with Hamiltonian $H(\sigma_\Lambda; \sigma_{\Lambda^c})$ is invariant under the Glauber dynamics. Moreover, for all σ_Λ and for all $x \in \Lambda$*

$$\mu(\sigma_\Lambda) c(x, \sigma_\Lambda) = \mu(\sigma^x_\Lambda) c(x, \sigma^x_\Lambda) \tag{2.46}$$

Finally, for all $f, g \in L^\infty(\mathcal{X}_\Lambda)$,

$$\mu(fLg) = \mu(gLf). \tag{2.47}$$

The preceding theorem is an important result, as it shows how “nicely” the Gibbs measure and Glauber dynamics work together. Precisely, the relation (2.46) is known as *detailed balance condition*. As we have already mentioned, spin dynamics extent the system out of equilibrium. However, according to Theorem 2.18, if the dynamics satisfy the detailed balance condition, one can understand how equilibrium is approached when a system is prepared in an out of equilibrium state. On top of that, the fact that the spin flip intensity satisfy (2.46) ensures that any initial spin state will eventually relax to the equilibrium thermodynamic equilibrium state for any non-zero temperature. Finally, detailed balance condition is such a strong property that makes the semigroup e^{Lt} self-adjoint and according to (2.47) measure μ is *reversible*.

2.5.3 Glauber dynamics with Kac potentials

In this subsection, we consider the Glauber dynamics with Kac potentials. We present the result that characterises the limiting behaviour of magnetisation. To state the result formally, we start by establishing some notation and setting up the problem.

Definition 2.19. Given $\beta > 0$ and $\gamma > 0$, we call Glauber dynamics the unique Markov process on $\mathcal{X}$ (see Sect. 2.1) whose generator is the operator L_γ

$$L_\gamma f(\sigma) = \sum_{x \in \mathbb{Z}} c_\gamma(x, \sigma)(f(\sigma^{(x)}) - f(\sigma))$$

Let Λ be any finite set of $\mathbb{Z}$ which contains x and the spin at x does not interact with the spins at Λ^c. Then, the flip rate is given

$$c_\gamma(x, \sigma) = \frac{1}{Z_\gamma(\sigma_{x^c})} e^{-\frac{\beta}{2}\Delta_x H_\gamma(\sigma)} = \frac{e^{-\beta h_\gamma(x)\sigma(x)}}{e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}}$$

where $\Delta_x H_\gamma(\sigma) = H_\gamma((\sigma^x)_\Lambda) - H_\gamma(\sigma_\Lambda)$ is the change of the energy due to the spin flip at x with $H_\gamma(\sigma_\Lambda)$ being the energy of a spin configuration σ_Λ

$$H_\gamma(\sigma_\Lambda) = -\frac{1}{2} \sum_{x,y \in \Lambda} J_\gamma(x, y)\sigma_\Lambda(x)\sigma_\Lambda(y) - h \sum_{x \in \Lambda} \sigma_\Lambda(x).$$

Its energy regarding also the interaction between spins in Λ and spins in its complement Λ^c is given,

$$H_\gamma(\sigma_\Lambda; \sigma_{\Lambda^c}) = H_\gamma(\sigma_\Lambda) - \sum_{x \in \Lambda} \sum_{x \in \Lambda^c} J_\gamma(x, y)\sigma_\Lambda(x)\sigma_{\Lambda^c}(y)$$

and

$$h_\gamma(x) = h + \sum_{y \neq x} J_\gamma(x, y)\sigma(y).$$

$J_\gamma(x, y)$ is defined in (2.32).

Notice that

$$\frac{c_\gamma(x, \sigma)}{c_\gamma(x, \sigma^x)} = e^{-\beta\Delta_x H_\gamma(\sigma)}$$

which means that the detailed balance condition holds. The space of realisations of the Glauber dynamics is $D(\mathbb{R}_+, \mathcal{X})$ the Skorohod space of cadlag trajectories, (continuous from the right and with limits from the left) and we denote the process by $\{\sigma_t\}_{t\geq 0}$ on $D(\mathbb{R}_+, \mathcal{X})$. By recalling the DLR equations in Sect. 2.4, we give the following definition:

Definition 2.20. The Gibbs measure $\mu_{\beta,h,\gamma}$ is any probability on $\mathcal{X}$ for which for every $x \in \mathbb{Z}$ and any σ, the probability that $\sigma(x) = \pm 1$ conditioned on the σ-algebra generated by all the spins $\sigma_{\mathbb{Z}\setminus x}$ is defined by

$$\mu_{\beta,h,\gamma}(\sigma(x) = \pm 1 | \{\sigma_{\mathbb{Z}\setminus x}\}) = \frac{e^{\pm\beta h_\gamma(x)}}{e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}}$$

We can check that the property (2.47) is satisfied for the Gibbs measure $\mu_{\beta,h,\gamma}$. This is the microscopic description of the problem, and the goal is to present its deterministic behaviour. The idea is that as $\gamma \to 0$, more and more spins feel the same potential, while each one individually has a random behaviour. However, in the limit the collectivity behaves deterministically, due to the law of large numbers. We call that limit *mesoscopic limit*, and we will say more about that in the next section. Let us then establish the notation in the mesoscopic scale that elucidates the separation between the microscopic and the mesoscopic scale. In general, we denote the mesoscopic points by $r, r', ...$ rather than by $x, y, ..$ as in the microscopic model, and $\mathbb{R}$ is the mesoscopic space. A mesoscopic point $r \in \mathbb{R}$ can be viewed on the shrunk lattice $\gamma\mathbb{Z}$ the following way: for every $r \in \mathbb{R}$, we associate a point

$$[r]_\gamma = \gamma x,$$

if $r \in [\gamma x, \gamma(x+1))$ with $x \in \mathbb{Z}$. Obviously, $[r]_\gamma \in \gamma\mathbb{Z}$. We then partition $\mathbb{R}$ into intervals $\{r : [r]_\gamma = \gamma x\}$. For example, if one wants to see a configuration σ in the mesoscopic level, let us denote it by $\sigma_\gamma(r)$, then the relation between σ and $\sigma_\gamma(r)$ is given by $\sigma_\gamma(r) = \sigma(x)$ when $x = \gamma^{-1}[r]_\gamma$. In general, we can view any measurable function defined on $\mathbb{Z}$ as a measurable function on $\mathbb{R}$: if $\mathcal{M}(X)$ is the set of all measurable functions defined on a measurable space X, we define $\Gamma_\gamma : \mathcal{M}(\mathbb{Z}) \to \mathcal{M}(\mathbb{R})$ such that

$$[\Gamma_\gamma(f)](r) = f(x), \text{ if } x = \gamma^{-1}[r]_\gamma$$

Definition 2.21. [Block spin transformation] For any $\alpha \in (0,1)$, for any $f \in \mathcal{M}(\mathbb{R})$ we define its block spin transformation, $f^{\gamma,\alpha} \in \mathcal{M}(\mathbb{R})$,

$$f^{(\gamma,\alpha)}(r) = \frac{\int \mathbb{1}_{|[r]_\gamma - [r']_\gamma| \leq \gamma^{1-\alpha}} f(r') dr'}{\int \mathbb{1}_{|[r]_\gamma - [r']_\gamma| \leq \gamma^{1-\alpha}} dr'}$$

From a microscopic point of view, it should be more convenient to use another version of this definition: for any $f \in \mathcal{M}(\mathbb{Z})$

$$[\Gamma_\gamma(f)]^{(\gamma,\alpha)}(r) = \frac{1}{|C_{\gamma^{-\alpha}}|} \sum_{y \in C_{\gamma^{-\alpha}}} f(y)$$

where $C_{\gamma^{-\alpha}} = \{y : |y - x| \leq \gamma^{-\alpha}\}$.

Having established both the miscroscopic and the mesoscopic description, we state the main result of the section that roughly says that the limiting magnetisation evolves under the deterministic, non-local evolution equation given in (3.10).

Theorem 2.22 ([27]). *For any $\alpha \in (0, 1)$ and $\zeta > 0$, there are a and b positive, and for any n and $k^* \geq 2$, there is c so that the following holds. For all γ small enough and, given γ, for all $\sigma \in \{-1, 1\}^{\mathbb{Z}}$ and $m \in \mathcal{M}(\mathbb{Z})$, $\|m\|_\infty \leq 1$ for which*

$$\sup_{|r| \leq k^* \gamma^{-1}} |(\sigma_\gamma)^{(\gamma,\alpha)}(r) - m^{(\alpha,\gamma)}(r)| \leq \gamma^\zeta$$

we have that

$$\mathbb{P}^\gamma_\sigma \left(\sup_{t \leq a \log \gamma^{-1}} \sup_{|r| \leq (k^*-1)\gamma^{-1}} \left|(\sigma_{\gamma,t})^{(\gamma,\alpha)}(r) - m^{(\alpha,\gamma)}(r, t)\right| > \gamma^b \right) \leq c\gamma^n$$

where $\mathbb{P}^\gamma_\sigma$ is the law of the Glauber dynamics when the process starts at time 0 from σ and

$$m^{(\gamma,\alpha)}(r, t) \equiv (m(\cdot, t))^{(\gamma,\alpha)} (r) \tag{2.48}$$

$m(\cdot, t)$ being the unique solution of the cauchy problem

$$\frac{d}{dt} m = -m + \tanh\{\beta(J * m + h)\}, \tag{2.49}$$

where $J * m(x, t) = \int_{\mathbb{R}} J(x - y)m(y, t)\, dy$.

2.6 Mesoscopic Theory

In this section, we discuss magnetic systems at a mesoscopic level. The mesoscopic theory focuses on free energy functionals of mesoscopic states, as well as characterisations of the equilibrium states. A state in the mescoscopic theory is a measurable function $m(r)$ on a region Λ or the whole $\mathbb{R}^d$. As we study magnetic systems, $m(r)$, plays the role of the magnetisation density at point r. At this point, two important issues arise and need further discussion: the first one is that the magnetisation in general is the average of spins on the underlying microscopic level, and this establishes the connection between the two scales. The second one concerns with the separation between the microscale and the mesoscale and consequently the connection of r with the microscopic units.

For a better understanding, it is worth looking at the scale through Kac potentials, that we have extensively analysed in this chapter. The idea of Kac potential is that by rescaling a function J of compact support, using a diverging factor γ^{-1}, with $\gamma > 0$ being small, while the integral is kept fixed, Kac potentials introduce a suitable mesoscopic scale, on which the local magnetisation is the quantity under investigation. This means that the size of the box that the average of spins have been considered in, defines the spatial scale of the mesoscopic scale in terms of the microscopic one. Thus, one could say that the mesoscopic scale is a level up from the microscopic scale. In Sect. 2.3 we remarked that the lattice spacing is assumed 1, then the range of Kac interactions is γ^{-1} and at last, the size of the region Λ, $|\Lambda|$, is of order ϵ^{-1} with $\epsilon > 0$ small such that $1 << \gamma^{-1} << \epsilon^{-1}$. One might see the separation between scales in a different way, namely the size of the region can be considered of order 1 (macro), the lattice spacing is then ϵ (micro) and the range of interaction should be $\gamma^{-1}\epsilon$ (meso). Obviously this renders the mesoscopic scale as intermediate scale between the microscopic and the macroscopic scale. Therefore, we think the magnetisation $m(r)$ as the average of spins on a microscopically large box around r.

2.6.1 The L-P functional

As remarked in Sect. 2.3.1, in the Ising model with Kac potentials the Gibbsian probability of observing averages of spins in particularly chosen cubes (coarse magnetisation) is given through the L-P functional. In this subsection, we isolate the functional from its underlying microscopic model and we present few properties of a functional with a structure as in (2.38), but without involving any microscopic information. Note that for this reason, we slightly change the notation, even for quantities that we have used so far.

We consider the free excess energy functional $\mathcal{F}_{\beta,h} : L^\infty(\mathbb{R}^d, [-1,1]) \to [0,\infty)$ which is given by

$$\mathcal{F}_{\beta,h}(m) = \int_{\mathbb{R}^d} \tilde{\phi}_{\beta,h}(m(r))dr + \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(r,r')[m(r)-m(r')]^2 dr dr', \tag{2.50}$$

where

$$\tilde{\phi}_{\beta,h}(s) = \tilde{\phi}_{\beta,h}(s) - \min_{|s|\le 1} \tilde{\phi}(s), \tag{2.51}$$

with $\tilde{\phi}_{\beta,h}(s)$ is given by (2.25) and J is non-negative functions, translational invariant, $J(r) = 0$ for all $|r| > 1$, $\int_{\mathbb{R}} J(r)dr = 1$ and $J \in C^2(\mathbb{R}^d)$.

The minimisers of $\mathcal{F}_{\beta,h}$ are the equilibrium states, which are also called *pure phases* of the system. Due to the second integral of r.h.s of (2.50), there are constant functions identically equal to the minimisers of $\tilde{\phi}_{\beta,h}$.

(a) For $h \neq 0$, $\tilde{\phi}_{\beta,h}$ has a unique minimiser $m_{\beta,h}$.

(b) For $h = 0$ and $\beta \leq 1$, $\tilde{\phi}_{\beta,h}$ has also a unique minimiser.

(c) For $h = 0$ and $\beta > 1$, $\tilde{\phi}_{\beta,h}$ has two minimisers, $\pm m_\beta$ where m_β satisfies the mean field equation (2.27).

(d) For $h = 0$ and any $\beta > 0$, there is no equilibrium state with magnetisation $m \in (-m_\beta, m_\beta)$.

(e) If $s \notin [-m_\beta, m_\beta]$, there exists a value h_s such that $m(r) = m_{\beta,h_s}$ is a unique minimiser of $\mathcal{F}_{\beta,h}$ and thus, m_{β,h_s} is an equilibrium state.

Furthermore, we define the excess free energy in bounded domains of $\mathbb{R}^d$. Let $m \in L^\infty(\mathbb{R}^d, [-1,1])$, Λ a Borel set in $\mathbb{R}^d$, then

$$\mathcal{F}(m) = \mathcal{F}_\Lambda(m_\Lambda | m_{\Lambda^c}) + \mathcal{F}_{\Lambda^c}(m_{\Lambda^c}), \tag{2.52}$$

where $\mathcal{F}_\Lambda(m_\Lambda | m_{\Lambda^c})$ is the excess free energy in Λ with boundary conditions m_{Λ^c} and $\mathcal{F}_\Lambda(m_\Lambda)$ the excess free energy in Λ, without interactions with Λ^c. Precisely,

$$\mathcal{F}_\Lambda(m_\Lambda) = \int_\Lambda \phi_{\beta,h}(m_\Lambda(r))dr + \frac{1}{4}\int_\Lambda\int_\Lambda J(r,r')[m_\Lambda(r) - m_\Lambda(r')]^2 dr dr',$$

$$\mathcal{F}_\Lambda(m_\Lambda | m_{\Lambda^c}) = \mathcal{F}_\Lambda(m_\Lambda) + \frac{1}{2}\int_\Lambda\int_{\Lambda^c} J(r,r')[m_\Lambda(r) - m_{\Lambda^c}(r')]^2 dr dr'$$

Finally, the free energy in bounded domains is given by:

$$\mathcal{F}_\Lambda(m_\Lambda | m_{\Lambda^c}) = F_\Lambda(m_\Lambda | m_{\Lambda^c}) - R_\Lambda(m_{\Lambda^c}) \tag{2.53}$$

with

$$F_\Lambda(m_\Lambda | m_{\Lambda^c}) = F_\Lambda(m_\Lambda) - \int_\Lambda\int_{\Lambda^c} J(r,r')m_\Lambda(r)m_{\Lambda^c}(r')]dr dr' \tag{2.54}$$

$$F_\Lambda(m_\Lambda) = -\int_\Lambda hm_\Lambda(r) - \frac{1}{\beta}\mathcal{S}(m_\Lambda(r)) - \frac{1}{2}\int_\Lambda\int_\Lambda J(r,r')m_\Lambda(r)m_\Lambda(r')]dr dr' \tag{2.55}$$

$$R_\Lambda(m_{\Lambda^c}) = -|\Lambda|\min_{|s|\leq 1}\tilde{\phi}(s) - \frac{1}{2}\int_\Lambda\int_{\Lambda^c} J(r,r')m^2_{\Lambda^c}(r')]dr dr' \tag{2.56}$$

When m_{Λ^c} is a fixed boundary condition, $R_\Lambda(m_{\Lambda^c})$ is a constant, and therefore the variational problems for $\mathcal{F}_\Lambda(m_\Lambda|m_{\Lambda^c})$ and $F_\Lambda(m_\Lambda|m_{\Lambda^c})$ are equivalent. Notice that $F_\Lambda(m_\Lambda|m_{\Lambda^c})$ is the L-P functional as defined in (2.38). We present two of the main properties of the functional, which are needed in a variational point of view: positivity and lower semi-continuity. The positivity of (2.50) is clearly true. About lower semi-continuity:

Theorem 2.23. *If $m_n \in L^\infty(\mathbb{R}^d, [-1,1])$ such that $\|m_n - m\|_{L^\infty(\Delta)} \to 0$ for any compact $\Delta \subset \mathbb{R}^d$ then*

$$\liminf_{n\to\infty} \mathcal{F}_{\beta,h}(m_n) \geq \mathcal{F}_{\beta,h}(m)$$

Let $\Lambda \subset \mathbb{R}^d$ be a bounded Borel set, $m_n \in L^\infty(\Lambda, [-1,1])$, $m_{\Lambda^c} \in L^\infty(\Lambda^c, [-1,1])$. Then if m_n converges weakly in $L^2(\Lambda)$ to m,

$$\liminf_{n\to\infty} \mathcal{F}_\Lambda(m_n|m_{\Lambda^c}) \geq \mathcal{F}_\Lambda(m|m_{\Lambda^c})$$

while if $m_n \to m$ almost everywhere $\mathcal{F}_\Lambda(m_n|m_{\Lambda^c}) \to \mathcal{F}_\Lambda(m|m_{\Lambda^c})$.

A last important comment for the L-P functional that is worth to stress, is its connection with the *Ginzburg-Landau functional*. Let us remind the functional before we go through the analysis (see [60]). The Ginzburg-Landau excess free energy functional

$$\mathcal{F}^{GL}(m) = \int_{\mathbb{R}^d} W(m(r)) + C|\nabla m(r)|^2 dr \tag{2.57}$$

is the simplest example for phase transition when the function W is a double-well potential. As before, the function $m(\cdot)$ is the magnetisation density. To minimise the functional $\mathcal{F}^{GL}$, the behaviour of each term intuitively goes as follows: the first integral of the r.h.s of (2.57) favours those m that are close to the minima of W while the second integral penalises variations of m. Therefore, the smaller $\mathcal{F}^{GL}(m)$, the closer to equilibrium profile.

We can view the functional (2.57) as a first-order expansion of a non-local functional (2.50): for simplicity let us consider $d = 1$, on slowly varying functions, that is $m(r) = m^*(\epsilon r)$, m^* being a smooth function independent of ϵ, then to the leading orders in ϵ as $\epsilon \to 0$, the second term in (2.50) we have the following

$$\frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} J(r,r')[m^*(\epsilon r) - m^*(\epsilon r')]^2 dr dr' = \epsilon^2 \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} J(r,r')|\nabla m^*(\epsilon r)|^2 (r-r')^2 dr dr'$$

By choosing $C = \frac{1}{2}\int_{\mathbb{R}} J(0,r) r^2 dr$, the two terms become equal, and with the choice $W(m) = \phi_{\beta,h}(m)$, where $\phi_{\beta,h}$ is the mean field free energy of the Ising model, the relation between (2.50) and (2.57) is apparent. As a last remark on this section, we would like to stress that when W is a double-well potential, Ginzburg-Landau theory provides the existence of a phase transition. On the other hand, when $\beta > 1$ and $h = 0$, $\phi_{\beta,0}$ has a double-well shape and the system exhibits a phase transition. Thus, thermodynamic properties that are based on this term appear to be the same for both functionals. As we will see in the next section, the evolution equations obtained in a sense from the corresponding functionals have also similar structures.

2.6.2 Non-local L-P evolution

According to thermodynamics, the free energy decreases with time. To carry this out in mesoscopic theory the evolution is defined as the gradient flow dynamics for a free energy functional:

$$\frac{dm}{dt} = -\frac{\delta \mathcal{F}}{\delta m}$$

where the r.h.s stands for the functional derivative of $\mathcal{F}$, and in general states that the velocity field is anti-parallel to the gradient. Indeed, it is easy to check that energy decreases. In our context (for $\mathcal{F} \equiv \mathcal{F}_{\beta,h}$ which is given in (2.50)), we get

$$\frac{dm}{dt} = J * m + h - \frac{1}{\beta}\operatorname{arctanh}(m). \tag{2.58}$$

At this point, we recall the result presented in Sect. 2.5.3. In Theorem 2.22, as $\gamma \to 0$, the particular coarse magnetisation converges to a profile which satisfies

$$\frac{dm}{dt} = -m + \tanh\{\beta[J * m + h]\}. \tag{2.59}$$

The two evolutions are similar as it is discussed in [8]. So far, we have seen the non-local evolution equation of infinite systems.

Remark 2.24. Note that the gradient flow dynamics for the Ginzburg-Landau functional (2.57) is the known *Allen-Cahn equation*,

$$\frac{\partial m}{\partial t} = C\Delta m - W'(m)$$

If we slightly modify the non-local evolution equation (2.58) by adding and subtracting m, then the term $J \star m - m$ plays the role of Δm whereas the term $-\frac{1}{2\beta}\operatorname{arctanh}(m) + m$ is comparable to $W'(m)$.

Similarly, we define the same non local evolution equation in bounded regions or Borel sets $\Lambda \subset \mathbb{R}^d$ and in this case we call them partial dynamics. Thus, for $m \in L^\infty(\mathbb{R}^d, [-1, 1])$, its restriction to Λ with m_{Λ^c} unchanged w.r.t time,

$$\frac{dm_\Lambda}{dt} = -m_\Lambda + \tanh\{\beta[J \star (m_\Lambda + m_{\Lambda^c}) + h]\}$$

The non-local evolution equations (2.58), (2.59) with an initial condition define a Cauchy problem, and the question that arises is if the two Cauchy problems have a solution (existence and uniqueness). By computing the integral form of the equations, we can prove in fact that both of the Cauchy problems have a solution. To be precise, given an initial condition $m(r, 0) = m(r)$ [resp. $m_\Lambda(r, 0) = m_\Lambda(r)$], the integral forms are

$$m(r,t) = e^{-t} m(r) + \int_0^t e^{-(t-s)} \tanh\{\beta[J * m(r,s) + h]\} ds \tag{2.60}$$

$$m_\Lambda(r,t) = e^{-t} m_\Lambda(r) + \int_0^t e^{-(t-s)} \tanh\{\beta[J * (m_\Lambda(r,s) + m_{\Lambda^c}) + h]\} ds \tag{2.61}$$

Theorem 2.25. *Let $m \in L^\infty(\mathbb{R}^d, [-1, 1])$, then there is a unique function $m(r, t) \in L^\infty(\mathbb{R}^d \times \mathbb{R}, [-1, 1])$ which satisfies* (2.60) *with $m(\cdot, 0) = m(\cdot)$, $m(r, t)$ is continuously differentiable in t and for every r solves* (2.59)*. Analogous statements hold for the partial dynamics.*

Theorem 2.26. *Let $T_t(\cdot)$ be the semigroup which solves* (2.59) *with initial condition m. Then for any $t > 0$, the functions $T_t(m) - e^{-t}m$ on $\mathbb{R}^d$ is differentiable and its gradient is uniformly bounded. Furthermore, for any positive integer k, T_t map $C^k(\mathbb{R}^d)$ into itself. Analogous statements hold for the corresponding semigroup $T_t^\Lambda(\cdot)$.*

Theorem 2.27. *Let u and v both be in $L^\infty(\mathbb{R}^d, [-1, 1])$, then for any $t > 0$,*

$$\|T_t(u) - T_t(v)\|_\infty \le e^{(\beta-1)t} \|u - v\|_\infty$$

and for any Borel set $\Lambda \subset \mathbb{R}^d$,

$$\|[T_t^\Lambda(u)]_\Lambda - [T_t^\Lambda(v)]_\Lambda\|_\infty \le e^{(\beta-1)t} \|u - v\|_\infty.$$

Definition 2.28. $v(r, t)$ is *sub-solution* of the Cauchy problem of (2.59) with initial condition $v(\cdot, 0)$, if for any $t > 0$

$$\sup_{s \le t} \|v(\cdot, s)\|_\infty < \infty, \qquad \sup_{s \le t} \left\| \frac{dv(\cdot, s)}{ds} \right\|_\infty < \infty$$

for all r and $t > 0$,

$$\frac{dv}{dt} \leq -v + \tanh\{\beta[J * v + h]\}$$

In a analogous way, *super-solution* is defined.

Theorem 2.29. [Comparison Theorem] *Let* $v(r,t)$*,* $w(r,t)$ *and* $m(r,t)$ *be respectively a sub-solution, a super-solution and the solution of* (2.59) *with initial values* $v(r,0) \leq m(r,0) \leq w(r,0)$*. Then for all* r *and* $t \geq 0$

$$v(r,t) \leq m(r,t) \leq w(r,t).$$

For further details and proofs of the preceding Theorems, we refer to [60].

Part I

Minimal Cost for the Macroscopic Motion of an Interface

Introduction

In recent years, there has been a significant effort to derive deterministic models describing two-phase materials and their dynamical properties, [44]. Furthermore, with the inclusion of stochastic effects [32] one can study richer phenomena such as dynamic transitions between local minima. This is an extension of ideas already developed in the Freidlin-Wentsell theory [37] on random perturbation of dynamical systems. Such effects, can be encoded to action functionals whose minimizers prescribe the optimal transition. However, the choice of the action functional is not straightforward. Motivated by this, we investigate the law that governs the power needed to force a motion of a planar interface between two different phases of a given ferromagnetic sample with a prescribed speed V, in other words, the optimal way an one dimensional macroscopic interface between two phases moves from an initial to a final position within a fixed time. As we have already mentioned in Chapter 1, the evolution of a macroscopic phase boundary can be related rigorously to a lattice model of Ising-spins with Glauber dynamics by a multi-scale procedure, see [27, 48]. First, a spatial scaling of the order of the (diverging) interaction range of the Kac-potential is applied to obtain a deterministic limit on the so-called mesoscale, which follows a nonlocal evolution equation, see [27, 19]. This equation is then rescaled diffusively to obtain the macroscopic evolution law, in this case motion by mean curvature. For an appropriate choice of the parameters both limits can be done simultaneously to obtain a macroscopic (and deterministic) evolution law for the phase boundary, in this case motion by mean curvature. It is natural to ask for the corresponding large deviations result, i.e., for the probability of macroscopic interfaces evolving differently from the deterministic limit law. This is particularly interesting when studying metastable phenomena of transitions from one local equilibrium to another as one needs to quantify such large deviations which cannot be captured by the deterministic evolution (for the present context of Glauber dynamics and Kac potential we also refer to [51]). For the first step, i.e., deviations from the limit equation on the mesoscale, this has been achieved by F. Comets, [18].

In Chapters 3 and 4, we extend this result and derive the probability of large deviations for the macroscopic limit evolution starting from the microscopic Ising-Kac model. The technical difficulties are related to the fact that almost all of the system will be in one of the two phases, i.e., contribute zero to the large deviations cost, while a deviation happens only at the interface. This means that the exponential decay rate of the probability of our events is smaller than the number of random variables involved. As a consequence of these difficulties, our final result holds in one dimension only (i.e. no curvature), while several partial results do not depend on the dimension. If we were to follow the technique used in [18] we would obtain errors which are either diverging in a further parabolic rescaling or they can not be explicitly quantified with respect to the small parameter. Therefore we use a different technique by introducing coarse-grained time-space-magnetization boxes and explicitly quantifying all possible transitions in the coarse-grained state space.

The large deviations functional provides the action functional we are after. This is a well developed idea also in the more general setting of nonequilibrium systems [5] and here we examine it in the context of reversible dynamics describing *macroscopic* interface motion. Furthermore, this connection to the underlying stochastic process is also insightful for calculating the minimizers. For example, in Chapter 3 we borrow concepts from statistical mechanics such as contours, free energy, local equilibrium which allow us to better understand the structure of the cost functional and hence reduce it in a simpler and more easily treatable form.

Let us explain more precisely the setting of Chapters 3 and 4. We fix a space-time (ξ, τ) scale (macroscopic) and we consider the particular example of an interface which is forced to move from a starting position $\xi = 0$ (at $\tau = 0$) to a final position $\xi = R$ within a fixed time T. If such a motion occurs with constant velocity, being $V = R/T$, linear response theory and Onsager's principle suggest that the power (per unit area) needed is given by V^2/μ, where μ is a mobility coefficient. Our goal is to verify the limits of validity of this law in a stochastic model of interacting spins which mesoscopically gives rise to a model of interfaces.

In [20] the same question has been studied starting with a model in the mesoscopic scale (x,t) and examining the motion of the interface in the macroscopic scale after a diffusive rescaling: $x = \epsilon^{-1}\xi$ and $t = \epsilon^{-2}\tau$, where ϵ is a small parameter eventually going to zero. The authors considered a non local evolution equation obtained as a gradient flow of a certain functional penalizing interfaces. An interface can be described as a non-homogeneous stationary solution of this equation, therefore in order to produce orbits where the interface is moving (i.e., non stationary) the authors included an additional external force. To select among all possible forces they considered as a *cost functional* an L^2-norm of the external force whose minimizer provides the best mechanism for the motion of the interface. However, in our case of starting from a microscopic model of spins, instead of postulating an action functional we actually derive it as a large deviations functional. Then, in order to find the best mechanism for the macroscopic motion of the interface one has to study its minimizers. This is addressed in Chapter 4 where we use a strategy closely related to the one in [20] but with the extra complication that the new functional turns out to give a softer penalization on deviating profiles than the L^2 norm considered in [20].

There is a significant number of works in the literature studying closely related problems, mostly in the context of the stochastic Allen-Cahn equation. In [52, 53, 58], the authors study a minimization problem over all possible "switching paths" related to the Allen-Cahn equation: The cost functional is the L^2-norm of the forcing in the Allen-Cahn equation, which is what one would heuristically expect if one could define the large deviations rate functional for the Allen-Cahn equation with space-time white noise. Their results deal with the meso-to-macro limit of those rate functionals, but do not connect these rigorously to a stochastic process on the microscale. On the other hand, the large deviations have been studied in [32, 47, 43]. Furthermore, combining the above results, the large deviations asymptotics under diffusive rescaling of space and time are obtained in [4] (see also the companion paper [3]): the authors consider coloured noise and take both the intensity and the spatial correlation length of the noise to zero while doing simultaneously the meso-to-macro limit. This double limit is similar in spirit with our work presented in Chapters 3 and 4, with the difference that our noise is microscopic and the "noise to zero" limit is replaced by a "micro-to-meso" limit. Therefore, we derive (and subsequently minimize) the large deviations action functional directly from a microscopic process, hence completing this program of connecting the three scales: microscopic (process), mesoscopic (equation) and macroscopic (sharp-interface). Note also that in a coarse-grained (almost mesoscopic) scale, we have an equation which is comparable to a non-local Allen-Cahn type equation with a noise which is a martingale generated by the microscopic noise of each spin. However, they state the large deviations principle directly in the Γ-limit while we only obtain quantitative estimates for the upper and lower bound which are valid in this macroscopic scale; hence it would be interesting as a future work to consider this analysis also in our case, maybe in higher dimensions as well. Some numerical results were also presented in [31].

Finally, in the stochastic Allen-Cahn one adds by hand a “mesoscopic" white-noise in one dimension, or a properly coloured noise in higher dimensions (for more details about the motivation see the introduction in [3]). The connection to the stochastic Allen-Cahn is particularly interesting also in view of the results [9, 59] connecting the fluctuations of this microscopic process to the stochastic Allen-Cahn equation in a critical regime.

Chapter 3

Large Deviations for the Macroscopic Motion of an Interface

3.1 The model and preliminary results

3.1.1 The microscopic model

Let $\Lambda = [-L, L]$ and $\mathcal{T} = [0, T]$ be the macroscopic space and time domain, respectively. For ϵ a small parameter we denote by $\Lambda_\epsilon = [-\epsilon^{-1}L, \epsilon^{-1}L]$ and $\mathcal{T}_\epsilon = [0, \epsilon^{-2}T]$ the corresponding mesoscopic domains. Choosing another small parameter γ, we consider the microscopic lattice system $\mathcal{S}_\gamma = \Lambda_\epsilon \cap \gamma\mathbb{Z}$, as viewed from the mesoscale. We consider

$$\epsilon \equiv \epsilon(\gamma) = |\ln \gamma|^{-a}, \tag{3.1}$$

for some $a > 0$ to be determined in Section 3.3.7. Let σ be the spin configuration $\sigma := \{\sigma(x)\}_{x \in \mathcal{S}_\gamma} \in \{-1, +1\}^{\mathcal{S}_\gamma}$. The spins interact via a Kac potential which depends on the parameter γ and has the form

$$J_\gamma(x, y) = \gamma J(x - y), \quad x, y \in \mathcal{S}_\gamma,$$

where J is a function such that $J(r) = 0$ for all $|r| > 1$, $\int_{\mathbb{R}} J(r)dr = 1$ and $J \in C^2(\mathbb{R})$. Given a magnetic field $h \in \mathbb{R}$, we define the energy of the spin configuration σ_Δ (restricted to a subdomain $\Delta \subset \mathcal{S}_\gamma$), given the configuration σ_{Δ^c} in its complement, by

$$\begin{aligned} H_{\gamma,h}(\sigma_\Delta; \sigma_{\Delta^c}) &= -h \sum_{x \in \Delta} \sigma_\Delta(x) - \frac{1}{2} \sum_{x \neq y \in \Delta} J_\gamma(x, y)\sigma_\Delta(x)\sigma_\Delta(y) \\ &\quad - \sum_{\substack{x \in \Delta \\ y \in \Delta^c}} J_\gamma(x, y)\sigma_\Delta(x)\sigma_{\Delta^c}(y). \end{aligned} \tag{3.2}$$

In $\mathcal{S}_\gamma$, we consider Neumann boundary conditions for the spins. The corresponding finite volume Gibbs measure is given by

$$\mu^{\bar{\sigma}}_{\beta,\Delta,\gamma,h}(d\sigma) = \frac{1}{Z_{\beta,\Delta,h}} e^{-\beta H_{\gamma,h}(\sigma;\bar{\sigma})}, \tag{3.3}$$

where β is the inverse temperature and $Z_{\beta,\Delta,h}$ the normalization (partition function). We introduce the Glauber dynamics, which satisfies the detailed balance condition with respect to the Gibbs measure defined above, in terms of a continuous time Markov chain (see also Sect. 2.5). Let $\lambda : \{-1,+1\}^{\mathcal{S}_\gamma} \to \mathbb{R}_+$ be a bounded function and $p(\cdot,\cdot)$ a transition probability on $\{-1,+1\}^{\mathcal{S}_\gamma}$ that vanishes on the diagonal: $p(\sigma,\sigma) = 0$ for every $\sigma \in \{-1,+1\}^{\mathcal{S}_\gamma}$. Consider the space

$$X = (\{-1,+1\}^{\mathcal{S}_\gamma}, \mathbb{R}_+)^{\mathbb{N}}, \tag{3.4}$$

endowed with the Borel σ-algebra that makes the variables $\sigma_n \in \{-1,+1\}^{\mathcal{S}_\gamma}$ and $\tau_n \in \mathbb{R}_+$ measurable. For each $\sigma \in \{-1,+1\}^{\mathcal{S}_\gamma}$, let P_σ be the probability measure under which (i) $\{\sigma_n\}_{n\in\mathbb{N}}$, is a Markov chain with transition probability p starting from σ and (ii) given $\{\sigma_n\}_{n\in\mathbb{N}}$, the random variables τ_n are independent and distributed according to an exponential law of parameter $\lambda(\sigma_n)$. Any realization of the process can be described in terms of the infinite sequence of pairs (σ_n, t_n) where $t_0 = 0$ and $t_{n+1} = t_n + \tau_n$ determining the state into which the process jumps and the time at which the jump occurs:

$$\{\sigma_t\}_{t\geq 0} \leftrightarrow ((\sigma_1,t_1)\,,\,(\sigma_2,t_2)\,,\,\ldots\,,(\sigma_k,t_k)\,,\,\ldots).$$

The space of realizations of the Glauber dynamics is also equivalent to the Skorohod space of cadlag trajectories (continuous from the right and with limits from the left). $D(\mathbb{R}_+, \{-1,+1\}^{\mathcal{S}_\gamma})$.

From [49] we have that for every P_σ the sequence (σ_n, t_n) is an inhomogeneous Markov chain with infinitesimal transition probability given by

$$P(\sigma_{n+1} = \sigma',\, t \leq t_{n+1} < t + dt \,|\, \sigma_n = \sigma,\, t_n = s) = p(\sigma,\sigma')\lambda(\sigma)e^{-\lambda(\sigma)(t-s)}\mathbf{1}_{\{t>s\}}dt. \tag{3.5}$$

The flip rate λ is given by

$$\lambda(\sigma) = \sum_{x\in\mathcal{S}_\gamma} c(x,\sigma)$$

and the transition probability by

$$p(\sigma,\sigma') = [\lambda(\sigma)]^{-1} \sum_{x\in\mathcal{S}_\gamma} c(x,\sigma)\mathbf{1}_{\sigma'=\sigma^x},$$

where σ^x is the configuration obtained from σ flipping the spin located at x. The flip rates $c(x,\sigma)$ for the single spin at x in the configuration σ are defined by

$$c(x,\sigma) = \frac{1}{Z_\gamma(\sigma_{x^c})} e^{-\frac{\beta}{2}\Delta_x H_\gamma(\sigma)}, \tag{3.6}$$

where

$$\Delta_x H_\gamma(\sigma) = H_\gamma(\sigma^x) - H_\gamma(\sigma) = 2\sigma(x) \sum_{y\neq x} J_\gamma(x,y)\sigma(y),$$

and

$$Z_\gamma(\sigma_{x^c}) = e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}, \quad h_\gamma(x) = \sum_{y\neq x} J_\gamma(x,y)\sigma(y).$$

For later use we also express the rates as:

$$c(x;\sigma) = F_{\sigma(x)}(h_\gamma(x)), \quad \text{where} \quad F_{\sigma(x)}(g) = \frac{e^{-\sigma(x)\beta g}}{e^{-\beta g} + e^{\beta g}}. \tag{3.7}$$

Note that the flip rate is bounded both from above and below:

$$c_m := \frac{e^{-2\beta\|J\|_\infty}}{e^{2\beta\|J\|_\infty} + e^{-2\beta\|J\|_\infty}} \leq c(x,\sigma) \leq \frac{e^{2\beta\|J\|_\infty}}{e^{2\beta\|J\|_\infty} + e^{-2\beta\|J\|_\infty}} =: c_M. \tag{3.8}$$

3.1.2 The mesoscopic model

For $x \in \mathcal{S}_\gamma$, we divide Λ_ϵ into intervals I_i, of equal length

$$|I_i| = |I| := |\ln\gamma|^{-b}, \quad i \in \mathcal{I} := \left\{-\left\lfloor \frac{\epsilon^{-1}L}{|I|}\right\rfloor, \dots, \left\lfloor \frac{\epsilon^{-1}L}{|I|}\right\rfloor - 1\right\}$$

for some $b > 0$ to be determined in Sect. 3.3.7. Denoting also by $I(x)$ the interval that contains the microscopic point $x \in \mathcal{S}_\gamma$, we consider the block spin transformation given by

$$m_\gamma(\sigma;\, x,t) = \frac{1}{\gamma^{-1}|I(x)|} \sum_{y\in I(x)\cap\mathcal{S}_\gamma} \sigma_t(y). \tag{3.9}$$

In the sequel we will also need to specify it by the index $i \in \mathcal{I}$ of the coarse cell, i.e., denote it by $m_\gamma(\sigma; i,t)$ or use a time independent version $m_\gamma(\sigma; i)$ as well.

In [27] (see also Sect. 2.5.3), it has been proved that as $\gamma \to 0$ the function $m_\gamma(\sigma;\, x,t)$ converges in a suitable topology to $m(x,t)$ which is the solution of the following nonlocal evolution equation

$$\frac{d}{dt} m = -m + \tanh\{\beta(J * m)\}, \tag{3.10}$$

where $J * m(x,t) = \int_{\mathbb{R}} J(x-y)m(y,t)\,dy$ and $J \in C^2(\mathbb{R})$ is even, $J(r) = 0$ for all $|r| > 1$, $\int_{\mathbb{R}} J(r)dr = 1$ and non increasing for $r > 0$. We also suppose $\beta > 1$.

Furthermore, this equation is related to the gradient flow of the free energy functional (see also Sect. 2.6.2)

$$\mathcal{F}(m) = \int_{\mathbb{R}} \phi_\beta(m)dx + \frac{1}{4}\int_{\mathbb{R}\times\mathbb{R}} J(x,y)[m(x)-m(y)]^2 dx\, dy, \tag{3.11}$$

where $\phi_\beta(m)$ is the "mean field excess free energy"

$$\phi_\beta(m) = \tilde{\phi}_\beta(m) - \min_{|s|\le 1}\tilde{\phi}_\beta(s), \qquad \tilde{\phi}_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta}\mathcal{S}(m), \qquad \beta > 1,$$

and $\mathcal{S}(m)$ is the entropy and it is given by (2.26): We also denote by

$$f(m) := \frac{\delta \mathcal{F}}{\delta m} = -J * m + \frac{1}{\beta}\operatorname{arctanh} m \tag{3.12}$$

the functional derivative of $\mathcal{F}$. Thus, the functional in (3.11) is a Lyapunov functional for the equation (3.10):

$$\frac{d}{dt}\mathcal{F}(m) = -\frac{1}{\beta}\int_{\mathbb{R}} (-\beta J * m + \operatorname{arctanh} m)(m - \tanh(\beta J * m))\, dx \le 0,$$

since the two factors inside the integral have the same sign. This structure will be essential in the sequel, e.g. in Theorem 3.1.

Concerning the stationary solutions of the equation (3.10) in $\mathbb{R}$, it has been proved that the two constant functions $m^{(\pm)}(x) := \pm m_\beta$, with $m_\beta > 0$ solving the mean field equation $m_\beta = \tanh\{\beta m_\beta\}$ are stationary solutions of (3.10) and are interpreted as the two pure phases of the system with positive and negative magnetization.

Interfaces, which are the objects of this thesis, are made up from particular stationary solutions of (3.10). Such solutions, called *instantons*, exist for any $\beta > 1$ and we denote them by $\bar{m}_\xi(x)$, where ξ is a parameter called the centre of the instanton. Denoting $\bar{m} := \bar{m}_0$, we have that

$$\bar{m}_\xi(x) = \bar{m}(x - \xi), \tag{3.13}$$

where the instanton $\bar{m}$ satisfies

$$\bar{m}(x) = \tanh\{\beta J * \bar{m}(x)\}, \quad x \in \mathbb{R}. \tag{3.14}$$

It is an increasing, antisymmetric function which converges exponentially fast to $\pm m_\beta$ as $x \to \pm\infty$, see e.g. [28] and Figure 3.1. There are also α and a positive so that

$$\lim_{x\to\infty} e^{\alpha x}\bar{m}'(x) = a, \tag{3.15}$$

see [26], Theorem 3.1. Moreover, any other solution of (3.14) which is strictly positive [respectively negative] as $x \to \infty$ [respectively $x \to -\infty$], is a translate of $\bar{m}(x)$, see [29].

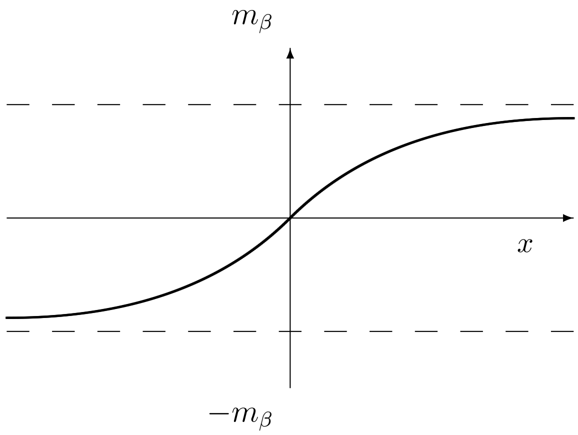

Figure 3.1: Instanton $\bar{m}$ with centre at 0.

Note also that in the case of finite volume $[-\epsilon^{-1}L, \epsilon^{-1}L]$ the solution $\bar{m}^{(\epsilon)}$ with Neumann boundary conditions is close to $\bar{m}$: for every $\epsilon > 0$ we consider the non-local mean field equation

$$m^{(\epsilon)} = \tanh\{\beta J^{\mathrm{neum}} * m^{(\epsilon)}\}, \quad |x| \le \epsilon^{-1}L, \tag{3.16}$$

where $m^{(\epsilon)} \in L^\infty([-\epsilon^{-1}L, \epsilon^{-1}L]; [-1,1])$ and

$$J^{\mathrm{neum}}(x,y) := J(x,y) + J(x, R_{\epsilon^{-1}L}(y)) + J(x, R_{-\epsilon^{-1}L}(y)),$$

with $R_l(y) := l - (y - l)$ being the reflection of y around l. By following [8], Section 3, or [6], Section 3.3, given $\zeta > 0$ there exists ϵ_0 such that for every $\epsilon < \epsilon_0$, there is $\bar{m}^{(\epsilon)}$ which is antisymmetric, solves (3.16), satisfies

$$\|\bar{m}^{(\epsilon)} - \bar{m}\|_{L^\infty([-\epsilon^{-1}L, \epsilon^{-1}L])} < \zeta \tag{3.17}$$

and it is unique in the above neighbourhood. See also [60], section 6.2.3.

Hence, if we start with an instanton, the evolution (3.10) will not move it. So, in order to impose a speed to the interface one has to add an external force to the equation (3.10). The result would be a deviation from (3.10) and any such deviation $\{\phi(x,t)\}_{x,t}$ corresponds to an external force that can produce it and which is given by

$$b(\phi)(x,t) := \dot{\phi}(x,t) + \phi(x,t) - \tanh(\beta J * \phi(x,t)), \tag{3.18}$$

where we have introduced the notation $\dot{\phi}(x,t) := \frac{d}{dt}\phi(x,t)$ and for b we explicit the dependence on ϕ. Later, when this dependence is not relevant we will only use b. Thus, such deviating profiles can be viewed as solutions of the following forced equation:

$$\frac{d}{dt}m = -m + \tanh(\beta J * m) + b, \qquad m(x,0) = m_0(x), \tag{3.19}$$

where the force term b is some prescribed function of x and t. In this thesis, we are interested in investigating the response of the system when imposing a mean velocity V to the front, i.e., we want to displace the interface from an initial position 0 to a final one, R, within a fixed time $T = R/V$. We consider two scales: the mesoscopic where the interface is diffuse and the macroscopic where the interface has a sharp jump, i.e., it is given by the step function $m_\beta(\mathbf{1}_{x\geq 0} - \mathbf{1}_{x<0})$. Let $[0,T]\times\mathbb{R}$ be the macroscopic time-space domain. After rescaling back to the mesoscopic variables we are interested in profiles in the set $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ where

$$\mathcal{U}[r,t] = \{\phi \in C^\infty(\mathbb{R}\times(0,t);(-1,1)) : \lim_{s\to 0^+}\phi(\cdot,s) = \bar{m}, \lim_{s\to t^-}\phi(\cdot,s) = \bar{m}_r\} \tag{3.20}$$

and where now in the mesoscopic variables the fronts are represented by the instantons $\bar{m}$ and $\bar{m}_r$. Due to the stationarity of $\bar{m}$, no element in $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ is a solution to the equation (3.10). Instead, to each element in $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ it corresponds an external force b as in (3.18), and in order to select among such forces one needs to introduce an appropriate action functional. In [20], the authors invoking linear response theory suggested the cost functional to be given by $\int_0^{\epsilon^{-2}T}\int_{\mathbb{R}} b(x,t)^2 dx\, dt$. In Chapter 4, instead of postulating the cost, we derive it directly from the underlying stochastic mechanism via large deviations over a certain class of functions. More precisely, to derive the cost from the stochastic dynamics we work in the space domain $[-\epsilon^{-1}L, \epsilon^{-1}L] \subset \mathbb{R}$ with Neumann boundary conditions. As it will be shown later, the main objects to which the cost concentrates are the instantons, which decay exponentially fast as $x \to \pm\infty$ and are well approximated by their finite volume counterparts as in (3.17). Hence, in order to avoid unnecessary technical complications we can concentrate here in the whole $\mathbb{R}$ and denote the new cost on $\mathbb{R}\times[0,\epsilon^{-2}T]$ by:

$$I_{[0,\epsilon^{-2}T]\times\mathbb{R}}(\phi) = \int_0^{\epsilon^{-2}T}\int_{\mathbb{R}} \mathcal{H}(\phi,\dot{\phi})(x,t)\, dx\, dt, \tag{3.21}$$

where for notational simplicity we neglect the dependence of the cost on $\mathbb{R}$. The density $\mathcal{H}(\phi,\dot{\phi})$ is given below and we will also denote it by $\mathcal{H}(x,t)$ in case we do not need to explicit the dependence on ϕ.

$$\begin{aligned}\mathcal{H}(\phi,\dot\phi) \quad := \quad & \frac{\dot\phi}{2}\left[\ln\frac{\dot\phi+\sqrt{(1-\phi^2)(1-\tanh^2(\beta J*\phi))+\dot\phi^2}}{(1-\phi)\sqrt{1-\tanh^2(\beta J*\phi)}}-\beta J*\phi\right] \\ & +\frac{1}{2}\Big[1-\phi\tanh(\beta J*\phi)- \\ & \qquad -\sqrt{(1-\phi^2)(1-\tanh^2(\beta J*\phi))+\dot\phi^2}\Big]. \end{aligned} \tag{3.22}$$

Properties of the cost functional

Given $(\phi,\dot\phi)$ we define

$$\begin{aligned} u &:= \phi \\ w &:= -\tanh(\beta J*\phi) \\ b &:= \dot\phi+\phi-\tanh(\beta J*\phi) \end{aligned}$$

and after a simple manipulation by a small abuse of notation we can write $\mathcal{H}$ as depending on (b,u,w) in the following form:

$$\begin{aligned}\mathcal{H}(b,u,w) \quad = \quad & \frac{1}{2}\Bigg\{(b-u-w)\log\frac{b-u-w+\sqrt{(b-u-w)^2+(1-u^2)(1-w^2)}}{(1-u)(1-w)} \\ & -\sqrt{(b-u-w)^2+(1-u^2)(1-w^2)}+1+uw\Bigg\}. \end{aligned} \tag{3.23}$$

The new functional, has a more complicated structure, but asymptotically has a similar behaviour: It is a straightforward calculation to see that uniformly on $u\in[-1,1]$ and $w\in(-1,1)$ we have:

$$\lim_{|b|\to\infty}\frac{\mathcal{H}(b,u,w)}{|b|\log(|b|+1)}=\frac{1}{2} \qquad \text{and} \qquad \lim_{|b|\to 0}\frac{\mathcal{H}(b,u,w)}{b^2}=\frac{1}{4(1+uw)}. \tag{3.24}$$

For further properties we refer the reader to [18]. In particular, in the sequel we will use the fact that

$$I_{\Lambda_\epsilon\times\mathcal{T}_\epsilon}(\phi)<\infty \qquad \text{iff} \qquad \dot\phi\ln|\dot\phi|,\,\dot\phi\ln\frac{1}{1-\phi}\mathbf{1}_{\{\dot\phi>0\}},\,\dot\phi\ln\frac{1}{1+\phi}\mathbf{1}_{\{\dot\phi<0\}}\in L^1(\Lambda_\epsilon\times\mathcal{T}_\epsilon). \tag{3.25}$$

Note that the cost assumed in [20] is approximating the case when b is small, but when b is large they are far from each other; hence it gives a stronger penalization of the deviating profiles than the one derived from the microscopic system. As we shall also see in the sequel, the minimizers will correspond to external fields b which are ϵ-small, so it is expected that the minimizers of the new functional will be the same with [20]. But still, we can not exclude a priori the cases that correspond to large external fields and this is a technical difficulty we have to overcome. Furthermore, we have a slightly different equation and a more complicated form of the cost. Thus, in this chapter, we find the minimizer of the derived cost $I_{[0,\epsilon^{-2}T]}(\phi)$ given in (3.21) over the class (3.20) following the strategy in [20] and adjusting the proof accordingly in order to overcome the aforementioned technical issues. To start with, we observe that the cost of a moving instanton with ϵ-small velocity (Figure 3.2), i.e.,

$$\phi_\epsilon(x,t) = \bar{m}_{\epsilon V t}(x), \quad V = \frac{R}{T},$$

is given by

$$I_{[0,\epsilon^{-2}T]}(\phi_\epsilon) = \frac{1}{4}\|\bar{m}'\|^2_{L^2(d\nu)} V^2 T,$$

where $\bar{m}'$ is the derivative of $\bar{m}$ and $\|\cdot\|_{L^2(d\nu)}$ denotes the L^2 norm on $(\mathbb{R}, d\nu(x))$ with $d\nu(x) = \frac{dx}{1-\bar{m}^2(x)}$. As in [20] it can be shown that other ways to move continuously the instanton are more expensive.

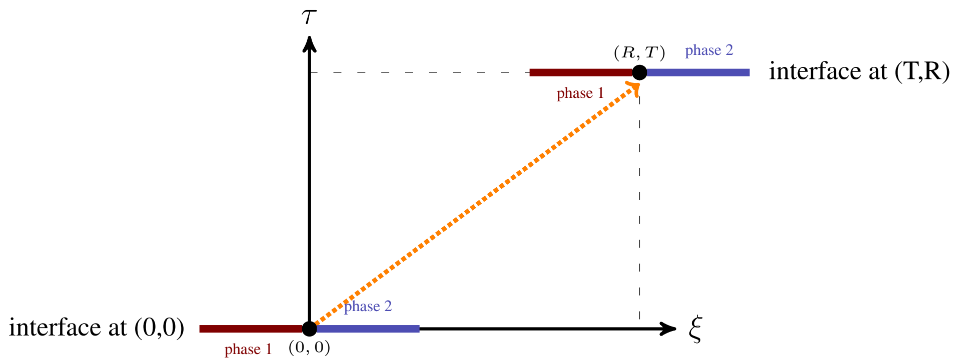

Figure 3.2: Macroscopic picture of moving instanton (orange dotted line) from 0 to R within fixed time T.

In such systems one can also observe the phenomenon of nucleations, namely the appearance of droplets of a phase inside another. In [6] and [7] it has been proved that for such a profile the cost is bounded by twice the free energy computed at the instanton:

Theorem 3.1. *For any $\vartheta > 0$ there is $\tau > 0$ and a function $\tilde{m}_{\epsilon,\tau}(x,s)$, $x \in \mathbb{R}$, $s \in [0, \tau\epsilon^{-3/2}]$, symmetric in x for each s and such that*

$$\tilde{m}_{\epsilon,\tau}(x,0) = m_\beta, \quad \tilde{m}_{\epsilon,\tau}(x,\tau\epsilon^{-3/2}) = \bar{m}_{\ell_\epsilon/2}(x),\ x \geq 0, \tag{3.26}$$

where $e^{-\alpha \ell_\epsilon} = \epsilon^{3/2}$*,* $\alpha > 0$ *as in* (3.15)*, and*

$$I_{\tau\epsilon^{-3/2}}(\tilde{m}_{\epsilon,\tau}) \leq 2\mathcal{F}(\bar{m}) + \vartheta. \tag{3.27}$$

Thus, if V gets large, there is a competition between the two values of the cost. Therefore, by creating more fronts we can make them move with smaller velocity with the gain in cost being larger than the extra penalty for the nucleations.

3.1.3 The macroscopic scale

This consists of the rescaled space-time domain $\Lambda \times \mathcal{T}$. The corresponding profiles are rescaled versions of the functions in the mesoscopic domain. In particular, the mesoscopically diffuse instanton is now a sharp interface between the two phases.

3.2 The problem and the main results

3.2.1 Large deviations at the macroscopic scale.

We consider an instanton initially at a macroscopic position 0 and move it to a final position R within a fixed time $T = R/V$, where V is a given value of the average velocity. At the mesoscopic scale functions that satisfy the above requirement are profiles in the set $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ defined by (3.20). Due to the stationarity of $\bar{m}$, no element in $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ is a solution of the equation (3.10). In order to produce such a motion, in [20] the authors considered an external force to the equation (3.10). Then, the optimal motion of the interface can be found by minimizing an appropriately chosen cost functional. Following their reasoning, given a profile $\phi(x,t)$ in (3.20) with time derivative $\dot{\phi}(x,t)$, we suppose that the profiles under investigation are solutions of the equation (3.19). As mentioned, in [20] the cost functional has been chosen to be $\int_0^{\epsilon^{-2}T} \|b(\cdot,t)\|^2_{L^2} dt$. In the present chapter we derive such an action functional by considering the underlying microscopic process and studying the probability of observing such a deviating event. Note that this is a large deviations away from a typical profile that satisfies the mesoscopic equation (3.10). The problem is formulated as follows: show that the probability of the event under investigation

$$\{\sigma_t : \sigma_0 \sim \bar{m}_0, \, \sigma_{\epsilon^{-2}T} \sim \bar{m}_{\epsilon^{-1}R}\}, \tag{3.28}$$

is logarithmically equivalent to the minimal cost computed over the class of functions $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ as $\gamma \to 0$. Here we are using the symbol $\sim$ to denote a suitable notion of distance that will be formally given below in Definition 3.2. In [18] the probability for the transition from the neighborhood of a stable equilibrium to another has been studied by establishing the equivalent to the Freidlin-Wentzell estimates, see [36]. The corresponding cost functional for $\mathbb{T} \times [0, T]$ is given by (3.21). However, in our case, we have to perform the same task but for the rescaled time and space domain $\Lambda_\epsilon \times \mathcal{T}_\epsilon$ in order to obtain a result which is valid also at the macroscopic scale. This is technically challenging as, in the case the time horizon as well as the volume scale with $\epsilon(\gamma)$, the error estimates providing (3.21) are not bounded when $\gamma \to 0$. To overcome it, we follow a different approach by coarse-graining the space of realizations of the process in all time, space and magnetization coordinates. Then, in order to calculate the probability of an event we intersect it with all possible coarse-grained "tubelets". The final result comes from an explicit calculation of the probability of such a tubelet and agrees with (3.21).

3.2.2 Main results

We divide $\Lambda_\epsilon \times \mathcal{T}_\epsilon \times [-1, 1]$ into space - time - magnetization boxes

$$I_i \times [j\Delta t, (j+1)\Delta t) \times [-1 + k\Delta, -1 + (k+1)\Delta),$$

where $i \in \mathcal{I} := \left\{ -\left\lfloor \frac{\epsilon^{-1}L}{|I|} \right\rfloor, \dots, \left\lfloor \frac{\epsilon^{-1}L}{|I|} \right\rfloor \right\}$, $j \in \mathcal{J} := \left\{ 0, 1, \dots, \left\lfloor \frac{\epsilon^{-2}T}{\Delta t} \right\rfloor - 1 \right\}$ and $k \in \mathcal{K}^\Delta := \{0, 1, \dots, \lfloor \frac{2}{\Delta} \rfloor - 1\}$. We choose the length to be

$$|I| = |\ln \gamma|^{-b}, \;\; \Delta t = \gamma^c, \; c < 1 \quad \text{and} \quad \Delta = \Delta t\, \eta_0, \tag{3.29}$$

respectively, where

$$\eta_0 \equiv \eta_0(\gamma) = |\ln \gamma|^{-\lambda_0}, \tag{3.30}$$

for some number $\lambda_0 > 0$ to be determined later in (3.89). Note that each I_i contains $\gamma^{-1}|I_i|$ many lattice sites of S_γ. Given such a coarse cell, we define the set of all *discretized paths* by

$$\bar{\Omega}_\gamma := \left\{ a \equiv \{a_{i,j}\}_{i \in \mathcal{I}, j \in \mathcal{J}} : \; a_{i,j} \in \mathcal{K}^\Delta \right\}. \tag{3.31}$$

Definition 3.2. Given $a \in \bar{\Omega}_\gamma$ and $\delta > 0$, recalling the definition of $m_\gamma(\sigma;\, x, t)$ in (3.9) for some $x \in I_i$, we say that $\sigma \in \{a\}_\delta$ if

$$\sup_{i \in \mathcal{I},\, j \in \mathcal{J}} |m_\gamma(\sigma;\, x, j\Delta t) - a_{i,j}| < \delta.$$

Given a function $m \in L^\infty(\Lambda_\epsilon \times \mathcal{T}_\epsilon)$, we say that $\sigma \in \{m\}_\delta$ if

$$\sup_{i \in \mathcal{I},\, j \in \mathcal{J}} \left| m_\gamma(\sigma;\, x, j\Delta t) - \frac{1}{|I_i|} \int_{I_i} m(x, j\Delta t)\, dx \right| < \delta.$$

Similarly, for a time-independent function $m \in L^\infty(\Lambda_\epsilon)$ we denote by $\sigma_t \in \{m\}_\delta$ (or $\{\sigma_t \sim m\}$ if we do not want to specify the parameter δ) the relation

$$\sup_{i \in \mathcal{I}} \left| m_\gamma(\sigma;\, x, t) - \frac{1}{|I_i|} \int_{I_i} m(x)\, dx \right| < \delta.$$

Given a set $A \subset D(\mathbb{R}_+, \{-1,+1\}^{\mathcal{S}_\gamma})$, to each $\sigma \in A$ we can associate an $a \in \bar{\Omega}_\gamma$ and a $\phi \in C^1(\Lambda_\epsilon \times \mathcal{T}_\epsilon)$ such that $\sigma \in \{a\}_\delta$ and $\sigma \in \{\phi\}_\delta$, respectively.

Definition 3.3. For $A \subset D(\mathbb{R}_+, \{-1,+1\}^{\mathcal{S}_\gamma})$, $\delta, \gamma > 0$, we define the sets

$$\bar{\Omega}_{\gamma,\delta}(A) := \{a \in \bar{\Omega}_\gamma \,:\, \exists \sigma \in A \;\text{ s.t. }\; \sigma \in \{a\}_\delta\} \tag{3.32}$$

and

$$\mathcal{U}_\delta(A) := \{\phi \in C^\infty(\Lambda_\epsilon \times \mathcal{T}_\epsilon) \,:\, \exists \sigma \in A \;\text{ s.t. }\; \sigma \in \{\phi\}_\delta\}. \tag{3.33}$$

The main result of this work are the following quantitative estimates:

Theorem 3.4. *For $\gamma > 0$ sufficiently small there exist $\delta_\gamma > 0$, $C_\gamma > 0$, $c_\gamma > 0$ such that the following holds:*

(i) For any closed set $C \subset D(\mathbb{R}_+, \{-1,+1\}^{\mathcal{S}_\gamma})$ and for $\gamma > 0$ small enough we have

$$\gamma \ln P(C) \le - \inf_{\phi \in \mathcal{U}_{\delta_\gamma}(C)} I_{\Lambda_{\epsilon(\gamma)} \times \mathcal{T}_{\epsilon(\gamma)}}(\phi) + C_\gamma, \tag{3.34}$$

with $\lim_{\gamma \to 0} C_\gamma = \lim_{\gamma \to 0} \delta_\gamma = 0$, where $\mathcal{U}_{\delta_\gamma}(C)$ is given in (3.33)*, $\epsilon(\gamma)$ is given by* (3.1) *and the cost functional $I_{\Lambda_{\epsilon(\gamma)} \times \mathcal{T}_{\epsilon(\gamma)}}(\phi)$ in* (3.21).

(ii) Similarly, for any open set $O \subset D(\mathbb{R}_+, \{-1,+1\}^{\mathcal{S}_\gamma})$ and for $\gamma > 0$ sufficiently small, we have that

$$\gamma \ln P(O) \ge - \inf_{\phi \in \mathcal{U}_{\delta_\gamma}(O)} I_{\Lambda_{\epsilon(\gamma)} \times \mathcal{T}_{\epsilon(\gamma)}}(\phi) + c_\gamma, \tag{3.35}$$

where again $\lim_{\gamma \to 0} c_\gamma = \lim_{\gamma \to 0} \delta_\gamma = 0$.

The above theorem is a quantitative version (for finite γ) of a Large Deviation Principle (LDP) for $\gamma^{-1}\epsilon^{-1}$ many random variables with a rate of only γ^{-1}. Note that if we wanted to write a statement directly in the limit $\gamma \to 0$ one should study the Γ-limit of the functional $I_{\Lambda_{\epsilon(\gamma)}\times\mathcal{T}_{\epsilon(\gamma)}}$, which might be a delicate issue since we need to express the limiting functional over singular functions and with the appropriate topology for the LDP to hold. However, we can find both the minimal value and the profiles to which it corresponds in the limit $\gamma \to 0$. This is the context of the next chapter, where we obtain a lower bound for the cost functional $I_{\Lambda_{\epsilon(\gamma)}\times\mathcal{T}_{\epsilon(\gamma)}}$ on the set of profiles in $\mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$, see (3.20). We start with a definition.

Definition 3.5. Given $R, T > 0$ and the mobility coefficient $\mu =: 4\|\bar{m}'\|_{L^2(d\nu)} > 0$, we define the cost corresponding to n nucleations and the related translations by

$$w_n(R,T) := n2\mathcal{F}(\bar{m}) + (2n+1)\left\{\frac{1}{\mu}\left(\frac{V}{2n+1}\right)^2 T\right\}, \tag{3.36}$$

where $V = R/T$, $\mathcal{F}$ is the free energy (3.11) and $\bar{m}$ the instanton, given in (3.14).

Note that the first term in (3.36) corresponds to the cost of n nucleations while the second to the cost of displacement of $2n+1$ fronts (with the smaller velocity $V/(2n+1)$, see Figure 3.3).

Theorem 3.6. *Let* $P > \inf_{n\geq 0} w_n(R,T)$.

(i) Then $\forall\gamma > 0$ *and for all sequences* $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ *with*

$$I_{\Lambda_\epsilon\times\mathcal{T}_\epsilon}(\phi_\epsilon) \leq P, \tag{3.37}$$

we have:

$$\liminf_{\epsilon\to 0} I_{\Lambda_\epsilon\times\mathcal{T}_\epsilon}(\phi_\epsilon) \geq \inf_{n\geq 0} w_n(R,T) - \gamma, \tag{3.38}$$

where $w_n(R,T)$ *is given in* (3.36).

(ii) There exists a sequence $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ *such that*

$$\limsup_{\epsilon\to 0} I_{\Lambda_\epsilon\times\mathcal{T}_\epsilon}(\phi_\epsilon) \leq \inf_{n\geq 0} w_n(R,T). \tag{3.39}$$

The proof of this theorem is given in Chapter 4.

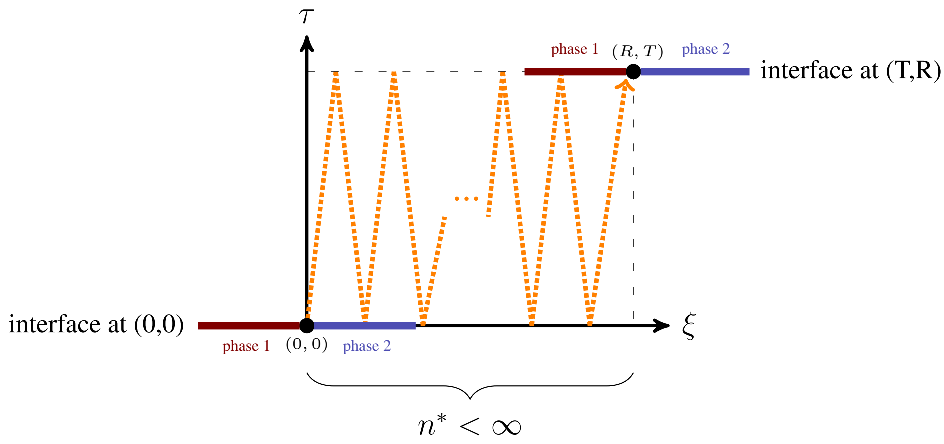

Figure 3.3: Macroscopic picture of nucleations (orange dotted line), where n^* is the optimal number of nucleactions which is given later in (4.27).

Combining the results in Theorem 3.4 and 3.6 we obtain a corollary about the optimal macroscopic motion of the interface. We start with some definitions: from the cost (3.36) we consider the set

$$\tilde{n}(R,T) := \operatorname{argmin} w_n(R,T) \tag{3.40}$$

which contains at most two elements. One can check that for certain values of R and T, n and $n+1$ nucleations have the same cost for some n, since we can get the same minimum value by one nucleation less, but higher velocity of the newly created fronts. Hence, the number of nucleations quantizes the cost. Now we define the set of profiles that have for some time $t \in \mathcal{T}_\epsilon$ at least the optimal number of nucleations. Given $\delta > 0$ we define the following set of mesoscopic paths

$$\mathcal{M}_{R,T}^{\delta,\epsilon} := \left\{ m \in L^\infty(\Lambda_\epsilon \times \mathcal{T}_\epsilon) : \min_{n \in \tilde{n}(R,T)} \left(\sup_{t \in \mathcal{T}_\epsilon} \mathcal{F}(m(\cdot,t)) - (2n+1)\mathcal{F}(\bar{m}) \right) > -\delta \right\}$$

and the set of realizations

$$A_\gamma^\delta := \left\{ \sigma : \; m_\gamma(\sigma;\cdot,\cdot) \in \mathcal{M}_{R,T}^{\delta,\epsilon(\gamma)} \right\}. \tag{3.41}$$

Note also that here we assume that the nucleations occur simultaneously as this is the most efficient way to do it, see Chapter 4. The fact that the instanton has travelled at least $\epsilon^{-1}R$ is represented by the set

$$C_\gamma^\delta := \{\sigma : \; m_\gamma(\sigma;\cdot,T) < \bar{m}_{\epsilon^{-1}R}(\gamma^{-1}\cdot) + \delta\}, \tag{3.42}$$

where $\bar{m}_{\epsilon^{-1}R}$ is given in (3.13). The following corollary states that if the transition happens, then it occurs through (at least) the optimal number of nucleations, i.e., the path leaves the level set of the free energy.

Corollary 3.7. *For any $\delta > 0$ and for the sets A^δ_γ and C^δ_γ defined in* (3.41) *and* (3.42) *we have:*

$$\lim_{\gamma \to 0} P_{\sigma_0}(A^\delta_\gamma | C^\delta_\gamma) = 1, \tag{3.43}$$

where P_{σ_0} denotes the law of the magnetization process starting at σ_0, with $\sigma_0 \in \{\bar{m}\}_\gamma$ as in (3.13).

The proof follows from the previous results. The key point is that if we consider the cost corresponding to the sets $(A^\delta_\gamma)^c \cap C^\delta_\gamma$ and C^δ_γ, by using the corresponding estimates from Theorem 3.4 for the closed and the open sets, we have that

$$\inf_{\phi \in \mathcal{U}_{\delta_\gamma}((A^\delta_\gamma)^c \cap C^\delta_\gamma)} I_{\Lambda_{\epsilon(\gamma)} \times \mathcal{T}_{\epsilon(\gamma)}}(\phi) - \inf_{\phi \in \mathcal{U}_{\delta_\gamma}(C^\delta_\gamma)} I_{\Lambda_{\epsilon(\gamma)} \times \mathcal{T}_{\epsilon(\gamma)}}(\phi) > 0,$$

since in the first set we do not include the optimal number of nucleations, hence the cost is higher than in the second. Then, the proof follows by applying the estimates of Theorem 3.4 to the conditional probability.

3.3 Proof of Theorem 3.4

3.3.1 Strategy of the proof of Theorem 3.4

Given a closed set $C \subset D(\mathbb{R}_+, \{-1,+1\}^{\mathcal{S}_\gamma})$ for Δ as in (3.29), consider the set $\bar{\Omega}_\gamma$. Now choose $\delta := \Delta/2$ and partition the sample space to get an upper bound by restricting to $\bar{\Omega}_{\gamma,\delta}(C)$, given in (3.32). Since we would like to work with smooth functions, we also define the following intermediate space:

Definition 3.8. We define by $\mathrm{PC}_{|I|}\mathrm{Aff}_{\Delta t}(\Lambda_\epsilon \times \mathcal{T}_\epsilon)$ the space of piecewise constant in space (in intervals of length $|I|$) and linear in time (in intervals of length Δt) functions. Given $a \in \bar{\Omega}_\gamma$, ϕ_a is the linear interpolation between the values $a(x,(j-1)\Delta t)$ and $a(x, j\Delta t)$):

$$\phi_a(x,t) := \sum_i \mathbf{1}_{I_i}(x) \sum_j \mathbf{1}_{[(j-1)\Delta t, j\Delta t)}(t) \left[\frac{a_{i,j} - a_{i,j-1}}{\Delta t} t + j \cdot a_{i,j-1} - (j-1) \cdot a_{i,j} \right]. \tag{3.44}$$

With the above choices we have:

$$\begin{aligned}
\gamma \ln P(C) &\leq \gamma \ln \sum_{a \in \bar{\Omega}_\gamma} P(\{a\}_\delta \cap C) \\
&\leq \sup_{a \in \bar{\Omega}_{\gamma,\delta}(C)} \left\{ -\sum_{i,j} \tilde{f}_{i,j}(a_{i,j}) \right\} + \gamma \ln |\bar{\Omega}_\gamma| \\
&\leq -\inf_{a \in \bar{\Omega}_{\gamma,\delta}(C)} I_{\Lambda_{\epsilon(\gamma)} \times \mathcal{T}_{\epsilon(\gamma)}}(\phi_a, \dot{\phi}_a) + C_\gamma \\
&\leq -\inf_{\phi \in \mathcal{U}_\delta(C)} I_{\Lambda_{\epsilon(\gamma)} \times \mathcal{T}_{\epsilon(\gamma)}}(\phi, \dot{\phi}) + C_\gamma, \qquad (3.45)
\end{aligned}$$

where $C_\gamma = \gamma \ln |\bar{\Omega}_\gamma|$. If we are able to find for a given tubelet $\{a\}_\delta$ an estimate of the form

$$\gamma \ln P(\{a\}_\delta) \leq \sum_{i,j} \tilde{f}_{i,j}(a_{i,j}) + C_\gamma. \qquad (3.46)$$

Here, $\tilde{f}_{i,j}(a)$ will be a discrete version of the density of the cost functional we are after.

In the second inequality we bounded the sum by the maximum value times its cardinality. Denoting by N_s, N_t and N_m the number of space, time and magnetization coarse cells, we have the following bound for the cardinality:

$$\left|\bar{\Omega}_\gamma\right| \leq N_m^{N_s \cdot N_t}, \quad \text{where} \;\; N_m \leq 2/\Delta. \qquad (3.47)$$

By using (3.29), this gives

$$\gamma \ln |\bar{\Omega}_\gamma| = \gamma \frac{\epsilon^{-3}}{\Delta t |I|} \ln \frac{2}{\Delta} \to 0, \qquad (3.48)$$

for all $c < 1$, as $\gamma \to 0$.

In order to prove (3.46), in Section 3.3.2 we divide $\mathcal{T}_\epsilon$ into time intervals with less (respectively more) spin flips than a fixed number. We call these time intervals good (respectively bad). We first show that the probability of having more than a given number (still diverging) of bad time intervals is negligible. In this way we partition the space of realizations by considering good and bad time intervals which we study separately. In each case we obtain a different form of $\tilde{f}$. In Section 3.3.3 we study the probability of the tubelet in a good time interval and by appropriately approximating it by a Poisson process for the number of positive and negative spin flips we obtain a formula for the density of the cost functional under the assumption that the fixed magnetization profiles a are far enough from their boundary values ± 1. This assumption will be removed later in Sect. 3.4.2 by showing that the probability of the process being close to any profile a can be approximated within some allowed error by the probability of the process being close to another profile $\tilde{a}$ as above. Another key step of the derivation of the cost in the good time intervals is to replace the random by deterministic rates and this is given in Section 3.3.5. Then, in Section 3.3.6 we treat the case of bad time intervals. More specifically we first show a rough upper bound for the probability in a given time interval which together with the estimated number of bad time intervals shows that the bad time intervals have vanishing contribution to the cost. We conclude with Section 3.3.7 where we prove that the discretized sum is a convergent Riemann sum yielding the cost functional we are after. To do that, we replace the discrete values a by the corresponding profile ϕ_a and subsequently obtain the cost functional over such functions given by $I_{\Lambda_{\epsilon(\gamma)}\times\mathcal{T}_{\epsilon(\gamma)}}(\phi_a)$ as in (3.21). Finally, in Lemma 3.19 we argue that it is enough to minimize over smoother versions of such functions, i.e., we will restrict our attention on the set given in (3.33). Once we have the upper bound we can look where the infimum occurs. Then for the lower bound we pick a collection $\{a^*_{i,j}\}_{i,j}$ which corresponds to the infimum and we bound the probability of an open set O by the probability of this particular profile, i.e.,

$$P(O) \geq P(\{a^*\}_\delta \cap O). \tag{3.49}$$

We skip the explicit proof of the lower bound as it is a straightforward repetition of the steps for obtaining the upper bound, with small alterations which will be discussed throughout the proof.

3.3.2 Too many jumps are negligible

We distinguish two types of time intervals, namely those with less (we call them *good*) or more (we call them *bad*) spin flips than a fixed number N to be a slightly larger number than the expected number of jumps within time Δt, i.e., we choose

$$N := \gamma^{-1}\epsilon^{-1}\Delta t\frac{1}{\eta_1}, \tag{3.50}$$

where

$$\eta_1 \equiv \eta_1(\gamma) := |\ln\gamma|^{-\lambda_1}, \tag{3.51}$$

for some $\lambda_1 > 0$ to be determined in (3.120). For the time interval $[j\Delta t, (j+1)\Delta t)$ we denote the number of jumps within this interval by:

$$N(\sigma_t, j) = \text{card}\,\{t \in [(j-1)\Delta t,\, j\Delta t) : \exists x \in \mathcal{S}_\gamma \text{ with} \lim_{\tau\to t^-}\sigma_\tau(x) = -\sigma_t(x)\}.$$

We decompose the path space X in (3.4) as follows:

$$X = \cup_{k \in \mathcal{J}} \cup_{j_1 < \ldots < j_k} D^{(k)}_{j_1,\ldots,j_k},$$

where

$$D^{(k)}_{j_1,\ldots,j_k} = \{N(\sigma_t, j) > N,\, j \in \{j_1, \ldots, j_k\} \text{ and } N(\sigma_t, j) \le N, \text{ otherwise}\}$$

is the set of realizations with k bad time intervals, indexed by $j_1, \ldots, j_k$. Then for the probability in the left hand side of (3.46) we have:

$$P(\{a\}_\delta) = P(\{a\}_\delta \cap \bar{D}_{\bar{k}}) + P(\{a\}_\delta \cap \bar{D}^c_{\bar{k}}),$$

where

$$\bar{D}_{\bar{k}} = \cup_{k > \bar{k}} \cup_{j_1 < \ldots < j_k} D^{(k)}_{j_1,\ldots,j_k}.$$

We select $\bar{k}$ such that $P(\{a\}_\delta \cap \bar{D}_{\bar{k}})$ is negligible. Note that

$$P_{\sigma_{(j-1)\Delta t}}(\{\sigma_t : N(\sigma_t, j) \ge N\}) \le e^{-cN \ln \frac{N}{\lambda \Delta t}}, \tag{3.52}$$

where $\lambda := \max_\sigma \lambda(\sigma)$. Therefore, given a configuration σ_0, we have

$$\begin{aligned} P_{\sigma_0}(\bar{D}_{\bar{k}}) &\le \sum_{k > \bar{k}} \binom{\frac{\epsilon^{-2}T}{\Delta t}}{k} (\sup_{\bar{\sigma}} P_{\bar{\sigma}}(N(\sigma_t, 1) > N))^k \\ &\le \sum_{k > \bar{k}} \left(\frac{\epsilon^{-2}T}{\Delta t} \exp\{-c\gamma^{-1}\epsilon^{-1}\Delta t \frac{1}{\eta_1} \ln \frac{1}{\eta_1}\} \right)^k \\ &\le e^{\bar{k}[\ln \frac{\epsilon^{-2}T}{\Delta t} - c\gamma^{-1}\epsilon^{-1}\Delta t \frac{1}{\eta_1} \ln \frac{1}{\eta_1}]}, \end{aligned} \tag{3.53}$$

which is negligible if we choose

$$\bar{k} := \frac{1}{\eta_2} \cdot \frac{1}{\epsilon^{-1}\Delta t \frac{1}{\eta_1} \ln \frac{1}{\eta_1}}, \tag{3.54}$$

for some

$$\eta_2 \equiv \eta_2(\gamma) = |\ln \gamma|^{-\lambda_2}, \quad \text{with} \quad \lambda_1 > \lambda_2 > 0, \tag{3.55}$$

so that $\eta_1 << \eta_2$, as required in Section 3.3.7, formula (3.104). Notice that $\bar{k} \to \infty$ as $\gamma \to 0$ since $\Delta t = \gamma^c$ while all other parameters grow logarithmically in γ.

Thus, overall we show that the probability of having too many bad time strips is negligible so for the upper bound we estimate it by the probability of the set $\{a\}_\delta \cap \bar{D}^c_{\bar{k}}$. We have:

$$P(\{a\}_\delta \cap \bar{D}^c_{\bar{k}}) = \sum_{k\le\bar{k}} \sum_{j_1<\ldots<j_k} \prod_{j\in\{j_1,\ldots,j_k\}} P_{\sigma_{j-1}}(\sigma_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N(\sigma_t, j) > N) \times \prod_{j\notin\{j_1,\ldots,j_k\}} P_{\sigma_{j-1}}(\sigma_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N(\sigma_t, j) \le N), \tag{3.56}$$

which can be further bounded by taking the cardinality $\bar{k}\binom{\frac{\epsilon^{-2}T}{\Delta t}}{\bar{k}}$ of the sum over k and $j_1 < \ldots < j_k$ and then the max over $(k, \{j_1, \ldots, j_k\})$. We call $k^*, \{j^*_1, \ldots, j^*_{k^*}\}$ the choice where the maximum is attained. On the good time strips ($j \notin \{j^*_1, \ldots, j^*_k\}$) we derive a discrete version of the density of the cost functional. On the other hand, on the bad time strips ($j \in \{j^*_1, \ldots, j^*_k\}$) we obtain upper and lower bounds and show that since these are few the corresponding cost is negligible. Note also that for the lower bound (3.49) we can simply restrict our attention on the good part D^c_0.

3.3.3 Good time intervals

In this section we compute the probability of the process being in a good time interval $[(j-1)\Delta t, j\Delta t)$.

Coarse-grained spin flip markov process $\{\bar{\sigma}_t\}_{t\ge 0}$

We establish a new spin flip markov process $\{\bar{\sigma}_t\}_{t\ge 0}$ which is defined on the same space and in a similar fashion as $\{\sigma_t\}_{t\ge 0}$, but does not distinguish among the spins of the same coarse cell I_i, $i \in \mathcal{I}$. The new transition probability is given by

$$\bar{P}(\sigma_{n+1} = \sigma',\, t \le t_{n+1} < t + dt \,|\, \sigma_n = \sigma,\, t_n = s) = \bar{p}(\sigma, \sigma')\bar{\lambda}(\sigma)e^{-\bar{\lambda}(\sigma)(t-s)}\mathbf{1}_{\{t>s\}}dt, \tag{3.57}$$

where $\bar{p}(\cdot,\cdot)$ and $\bar{\lambda}$ are given below. Recalling the coarse-graining over space intervals I_i, $i \in \mathcal{I}$, we first define the coarse-grained interaction potential

$$\bar{J}_\gamma(i, i') := \frac{1}{\gamma^{-2}|I|^2} \sum_{x\in I_i,\, y\in I_{i'}} J_\gamma(x, y), \quad \text{where } i, i' \in \mathcal{I}, \tag{3.58}$$

with $\bar{J}_\gamma(i, i) \equiv \bar{J}_\gamma(0) := \frac{1}{\gamma^{-1}|I|(\gamma^{-1}|I|-1)} \sum_{x,y\in I_i, x\ne y} J_\gamma(x, y)$. Note also that for all $x \in I_i$ and $y \in I_{i'}$ we have the bound:

$$|J_\gamma(x, y) - \bar{J}_\gamma(i, i')| \le \gamma|I|\|J'\|_\infty \mathbf{1}_{|x-y|\le 1}\mathbf{1}_{|i-i'|\le|I|^{-1}}. \tag{3.59}$$

The coarse-grained rates for $x \in I_i$ are given by

$$\bar{c}^i(x,\sigma) := \mathbf{1}_{x\in I_i}(x) F_{\sigma(x)}(\bar{h}_\gamma(x)), \tag{3.60}$$

where

$$\bar{h}_\gamma(x) = \mathbf{1}_{x\in I_i}(x) \sum_{i'\neq i} \bar{J}_\gamma(i,i') \sum_{y\in I_{i'}} \sigma(y) + \bar{J}_\gamma(i,i) \sum_{y\in I_i} \sigma(y). \tag{3.61}$$

and $F_{\sigma(x)}$ is given by (3.7). Then, the flip rate $\bar{\lambda}$ and the transition probability are respectively given by

$$\bar{\lambda}(\sigma) = \sum_{i=1}^{\epsilon^{-2}T/|I|} \sum_{x\in I_i} \bar{c}^i(x,\sigma), \quad \bar{p}(\sigma,\sigma') = [\bar{\lambda}(\sigma)]^{-1} \sum_{i=1}^{\epsilon^{-2}T/|I|} \sum_{x\in I_i} \bar{c}^i(x,\sigma)\mathbf{1}_{\sigma'=\sigma^x}.$$

In the next lemma we compare the processes σ and $\bar{\sigma}$:

Lemma 3.9. *For any $a \in \bar{\Omega}_\gamma$ there exists $c > 0$ such that for $\gamma > 0$ small enough*

$$e^{-\beta 2cL\epsilon^{-1}\gamma^{-1}\Delta t \frac{1}{\eta_1} C^*(\gamma)} \leq \frac{P_{\sigma_{(j-1)\Delta t}}(\sigma_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j \leq N)}{\bar{P}_{\sigma_{(j-1)\Delta t}}(\bar{\sigma}_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j \leq N)} \leq e^{\beta 2cL\epsilon^{-1}\gamma^{-1}\Delta t \frac{1}{\eta_1} C^*(\gamma)}, \tag{3.62}$$

where η_1 is given in (3.51) *and $C^*(\gamma) = |I| \|J'\|_\infty + \gamma \|J\|_\infty$.*

Remark 3.10. Note that after taking $\gamma \ln()$ and considering all time intervals, the error in (3.62) is negligible as $\frac{\epsilon^{-2}}{\Delta t}\epsilon^{-1}\frac{1}{\eta_1}\Delta t C^*(\gamma) \to 0$, as $\gamma \to 0$, if we choose

$$3a + \lambda_1 - b < 0. \tag{3.63}$$

Proof. We compare the rates of the processes σ_t and $\bar{\sigma}_t$: for any $x \in I_i$ from (3.59) and the properties of F in (3.7), starting from the same configuration σ' we have that there exists $c > 0$ such that

$$\begin{aligned} |c(x,\sigma')| - \bar{c}^i(x,\sigma')| &\leq c|h_\gamma(x) - \bar{h}_\gamma(x)| \\ &\leq c\Big| \sum_{y\neq x} J_\gamma(x,y)\sigma'(y) - \sum_{k\neq i} \bar{J}_\gamma(k,i) \sum_{y\in I_k} \sigma'(y) - \bar{J}_\gamma(0) \sum_{y\in I_i, y\neq x} \sigma'(y) \Big| \\ &\leq c\Big(\sum_{k\neq i}\sum_{y\in I_k} \big|J_\gamma(x,y) - \bar{J}_\gamma(k,i)\big| + \sum_{y\in I_i,\, y\neq x} \big|J_\gamma(x,y) - \bar{J}_\gamma(0)\big| \Big). \end{aligned} \tag{3.64}$$

Using (3.59) we obtain the error

$$|c(x,\sigma') - \bar{c}^i(x,\sigma')| \leq c\beta(|I|\|J'\|_\infty + \gamma\|J\|_\infty) =: c\beta C^*(\gamma), \tag{3.65}$$

which further gives that

$$|\lambda(\sigma') - \bar{\lambda}(\sigma')| \leq 2cL\epsilon^{-1}\gamma^{-1}\beta C^*(\gamma). \tag{3.66}$$

Replacing it by the Radon-Nikodym derivative between the laws of the processes σ_t and $\bar{\sigma}_t$ (see e.g. [49], Appendix 1, Proposition 2.6)

$$\left.\frac{dP}{d\bar{P}}\right|_{\mathcal{F}_t} = \exp\left\{ \int_0^t [\lambda(\sigma_s) - \bar{\lambda}(\sigma_s)]ds - \sum_{s\leq t} \ln \frac{\lambda(\sigma_{s^-})p(\sigma_{s^-},\sigma_s)}{\bar{\lambda}(\sigma_{s^-})\bar{p}(\sigma_{s^-},\sigma_s)} \right\}, \tag{3.67}$$

we obtain the upper bound $\gamma^{-1}\epsilon^{-1}C^*(\gamma)\Delta t$ for the integral in (3.67) and $NC^*(\gamma)$, with N as in (3.50) for the sum, which further yield the bounds of (3.62). $\square$

Let $\bar{L}$ be the generator of the new process $\{\bar{\sigma}_t\}_{t\geq 0}$. We consider the magnetization density at each coarse cell I_i of the new process $\{\bar{\sigma}_t\}_{t\geq 0}$

$$m_\gamma(\bar{\sigma}; i, t) := \frac{1}{\gamma^{-1}|I|} \sum_{x\in I_i} \bar{\sigma}_t(x),$$

as in (3.9) and (with slight abuse of notation) define

$$m_\gamma(\bar{\sigma}) \equiv \{m_\gamma(\bar{\sigma}; i)\}_{i\in\mathcal{I}}. \tag{3.68}$$

We are interested in the action of the generator on functions $f \in L^\infty(X)$ which are constant on the level sets $\{\bar{\sigma} \in X : m_\gamma(\bar{\sigma}; i) = m_i \in M, \forall i \in \mathcal{I}\}$. Note that such functions have the property that $f(\bar{\sigma}) = g(m_\gamma(\bar{\sigma}))$, for some $g \in L^\infty(M^{\mathcal{I}})$ and $M := \{-1, -1 + \frac{2}{\gamma^{-1}|I|}, \ldots, 1 - \frac{2}{\gamma^{-1}|I|}, 1\}$. Then there is a Markov generator $\mathcal{L}$ on $L^\infty(M^{\mathcal{I}})$ such that for any $g \in L^\infty(M^{\mathcal{I}})$ and any $\bar{\sigma} \in X$

$$e^{\bar{L}t} f(\bar{\sigma}) = e^{\mathcal{L}t} g(m_\gamma(\bar{\sigma})), \tag{3.69}$$

where $f(\bar{\sigma}) = g(m_\gamma(\bar{\sigma}))$. This is easy to show: we first denote the new coarse-grained process by $m(t) \equiv \{m_i(t)\}_{i\in\mathcal{I}}$ whose generator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}g(m) = \gamma^{-1}|I| \sum_i \Bigg(& \bar{c}_+(i,m) \left[g(m_i - \frac{2}{\gamma^{-1}|I|}) - g(m_i) \right] + \\ & + \ \bar{c}_-(i,m) \left[g(m_i + \frac{2}{\gamma^{-1}|I|}) - g(m_i) \right] \Bigg), \end{aligned} \tag{3.70}$$

with rates:

$$\bar{c}_\pm(i,m) := \frac{1 \pm m_i}{2} F_\mp(\bar{h}(i;m)), \tag{3.71}$$

where, by a slight abuse of notation compared to (3.61),

$$\bar{h}(i;m) := \gamma^{-1}|I| \sum_{i' \neq i} \bar{J}_\gamma(i,i')m_{i'} + \gamma^{-1}|I|\bar{J}_\gamma(0)m_i \tag{3.72}$$

and

$$F_\mp(h) = \frac{e^{\mp\beta h}}{e^{-\beta h} + e^{\beta h}}. \tag{3.73}$$

When $f(\bar{\sigma}) = g(m_\gamma(\bar{\sigma}))$ then $\bar{L}f(\bar{\sigma}) = \mathcal{L}g(m_\gamma(\bar{\sigma}))$. By induction on n, we have that $\bar{L}^n f(\bar{\sigma}) = \mathcal{L}^n g(m_\gamma(\bar{\sigma}))$ and expanding $e^{\bar{L}t}f$ in a power series, we obtain (3.69).

3.3.4 Poisson process for the jumps

To compute the probability for the coarse-grained process we realize the coarse-grained Glauber dynamics by constructing for each m_i two independent Poisson processes, $\underline{t}^i_\pm(m_i) := \{t^i_{\pm,1}(m_i) \leq \ldots \leq t^i_{\pm,n}(m_i) \leq \ldots\}$ called “random times" and then taking the product over all $m_i \in M$ and all $i \in \mathcal{I}$. Hence, we can construct the process $m(t) := \{m_i(t)\}_{i\in\mathcal{I}}$, $t \geq 0$, as follows: if at time $s \geq 0$ the process is in m then it remains in m until the minimum between the times $t^i_\pm := \min_{n\in\mathbb{N}}\{t^i_{\pm,n}(m_i)\}$ and over all i occurs. Then, for that i, the magnetisation m_i increases (respectively decreases) by $\frac{2}{\gamma^{-1}|I|}$. The case $\min_i t^i_- = \min_i t^i_+$ has probability 0.

3.3.5 From random to deterministic rates

The complication in the construction of $m(t)$ resides on the fact that we need to know how the random times are interrelated. Furthermore, the values of m_i and m_j (at the two coarse-grained boxes I_i and I_j, respectively) are correlated via the interaction potential $\bar{J}_\gamma$. Hence, for both of the above reasons, the analysis would become much simpler if we made the intensities of the random times independent of the current value m_i. To this end, we make them depend on some deterministic value of the profile which remains close to m_i during the whole time interval of length Δt. As a result, there will be only two rates for each $i \in \mathcal{I}$: one for the plus jumps and the other for the minus jumps. Let $N^\pm_{i,j-1}$ be the number of plus/minus random times during the time interval $[(j-1)\Delta t, j\Delta t]$ that occur in the i-th space interval. Note that for simplicity in the notation, in $N^\pm_{i,j-1}$ we do not carry the dependence on Δt. Then, the change of the magnetization in any time interval $[(j-1)\Delta t, j\Delta t)$ is equal to $2(N^-_{i,j-1} - N^+_{i,j-1})$. To formulate this idea we introduce new deterministic rates depending on the fixed configuration $a \equiv \{a_{i,j}\}_{i,j}$:

$$\bar{c}_{\pm}(i,a) := \bar{k}_{\pm}^{i,j-1}(a_{i,j-1}) F_{\mp}\left(\frac{1}{|I|}\int_{I_i} dr\, J * a_{j-1}(r)\right), \tag{3.74}$$

where $F_{\mp}$ is given in (3.73),

$$a_{j-1}(r) := \sum_{k\in\mathcal{I}} \mathbf{1}_{I_k}(r) a_{k,j-1}, \quad r \in \mathbb{R}, \quad j \in \mathcal{J}$$

and

$$\bar{k}_{\pm}^{i,j-1}(x) := \frac{1 \pm x}{2}. \tag{3.75}$$

Our goal is to use the distribution of the random variable $2(N_{i,j-1}^{-} - N_{i,j-1}^{+})$. More precisely, in Lemma 3.11 below, we show that the law of two independent Poisson processes with deterministic intensities $\gamma^{-1}|I|\bar{c}_{\pm}(i,a)$ is close to the law of two independent Poisson processes with intensities $\gamma^{-1}|I|\bar{c}_{\pm}(i, m((j-1)\Delta t))$.

By approximating the mean field process considering constant intensities

$$\gamma^{-1}|I|\bar{c}_{\pm}(i,a)$$

(one for the plus and one for the minus species), the resulting process is independent in each space box indexed by $i \in \mathcal{I}$. The Poisson probability of the occurrence of n random times at a given space box within a time interval of length Δt is given by

$$\mathbb{P}_{\gamma^{-1}|I|\bar{c}_{\pm}(i,a)}(N_{i,j-1}^{\pm} = n) = e^{-\gamma^{-1}|I|\bar{c}_{\pm}(i,a)\Delta t} \frac{(\gamma^{-1}|I|\bar{c}_{\pm}(i,a)\Delta t)^n}{n!}. \tag{3.76}$$

Given $d_{i,j-1} = \frac{a_{i,j}-a_{i,j-1}}{\Delta t} \in \mathbb{R}$ we consider the following event

$$B_{i,j-1}^{\delta}(a) := \left\{ \left| \frac{2}{\gamma^{-1}|I|}(N_{i,j-1}^{-} - N_{i,j-1}^{+}) - d_{i,j-1}\Delta t \right| < \delta, N_{i,j} \le N \right\}, \tag{3.77}$$

where the random variable $N_{i,j}$ stands for the number of jumps within the time interval $[(j-1)\Delta t, j\Delta t)$ in the space interval I_i.

Lemma 3.11. *Let $\nu^i = \mathbb{P}_{\gamma^{-1}|I|\bar{c}_{+}(i,a)} \times \mathbb{P}_{\gamma^{-1}|I|\bar{c}_{-}(i,a)}$ be the law of the product of two independent Poisson processes with intensities $\gamma^{-1}|I|\bar{c}_{+}(i,a)$ and $\gamma^{-1}|I|\bar{c}_{-}(i,a)$, respectively. Then, for any configuration $a \in \bar{\Omega}_\gamma$ and $\delta > 0$, we have that*

$$P_{m((j-1)\Delta t)}(m(j\Delta t) \in \{a_{\cdot,j}\}_{\delta}, N_j \le N) \le e^{2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)} \times$$
$$\times \prod_{i\in\mathcal{I}} \nu^i_{m_i((j-1)\Delta t)}(B_{i,j-1}^{\delta}(a)) \tag{3.78}$$

and

$$P_{m((j-1)\Delta t)}(m(j\Delta t) \in \{a_{\cdot,j}\}_\delta, N_j \le N) \ge e^{-2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)} \times$$
$$\times \prod_{i\in\mathcal{I}} \nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a)), \quad (3.79)$$

where $C^*(\gamma)$ is given in (3.65)*, η_1 in* (3.51) *and $B^\delta_{i,j-1}$ in* (3.77) *with $d_{i,j-1} = \frac{a_{i,j}-a_{i,j-1}}{\Delta t}$. Moreover, we denote by $\nu^i_{m_i((j-1)\Delta t)}(\cdot)$ the conditional probability of an event which starts from $m_i((j-1)\Delta t)$ at time $(j-1)\Delta t$.*

Remark 3.12. Finally, note that the error (that is, the $\ln$ of the factor in front of the r.h.s. of (3.79)) is negligible for the choice $\delta \equiv \delta_\gamma = \frac{\Delta}{2}$ with Δ as in (3.29), since, after considering all time intervals,

$$\frac{\epsilon^{-2}}{\Delta t}\epsilon^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta_\gamma) \to 0, \qquad \text{when } \gamma \to 0,$$

under the requirement (3.63) and the fact that Δt (in δ_γ) is a power of γ.

Proof. We consider a process $\{\bar m(t)\}_{t\ge 0}$ whose rates are constant and equal to $\gamma^{-1}|I|\bar c_\pm(i,a)$ as in (3.74). By comparing the rates $\bar c_\pm(i,m)$ and $\bar c_\pm(i,a)$ given in (3.71) and (3.74), respectively, we have:

$$|\bar c_\pm(i,m) - \bar c_\pm(i,a)| \le$$

$$\begin{aligned}
&\le c\delta + c\left|\gamma^{-1}|I|\sum_{k\ne i}\bar J_\gamma(i,k)m_k + \gamma^{-1}|I|\bar J_\gamma(0)m_i - \frac{1}{|I|}\int_{I_i} dr\, J * a_{j-1}(r)\right| \\
&\le c\delta + c\gamma^{-1}|I|\frac{1}{|I|}\int_{I_i} dr \sum_{k\ne i} a_k \frac{1}{|I|}\int_{I_k} dr' |J_\gamma(r,r') - \bar J_\gamma(i,k)| + \\
&\quad c\gamma^{-1}|I|\sum_{k\ne i}\bar J_\gamma(i,k)(a_k - m_k) + \gamma^{-1}|I|\frac{1}{|I|^2}\int_{I_i\times I_i} dr\, dr' |J_\gamma(r,r') - \bar J_\gamma(0)| \\
&\le c\delta + c(\gamma^{-1}|I|\frac{1}{|I|}\gamma|I|\|J'\|_\infty + \delta + \gamma\|J\|_\infty), \qquad (3.80)
\end{aligned}$$

where we have used (3.59) for the slightly different case, namely when $r, r' \in \mathbb{R}$ rather than just on $\mathcal{S}_\gamma$. Recalling $C^*(\gamma)$ from (3.65), we obtain:

$$|\lambda(m) - \bar\lambda(\bar m)| \le 2cL\epsilon^{-1}\gamma^{-1}\beta(C^*(\gamma)+\delta). \qquad (3.81)$$

By using (3.67) we get

$$\begin{aligned}
e^{-2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)} &\le \frac{P_{m(j-1)\Delta t}(m(j\Delta t) \in \{a_{\cdot,j}\}_\delta, N_j \le N)}{P_{m(j-1)\Delta t}(\bar m(j\Delta t) \in \{a_{\cdot,j}\}_\delta, N_j \le N)} \le \\
&\le e^{2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)}. \qquad (3.82)
\end{aligned}$$

Furthermore, since the processes $\bar{m}_i$ are independent with respect to $i \in \mathcal{I}$, we can write (3.82) in the following form:

$$
\begin{aligned}
P_{m(j-1)\Delta t}\left(m(j\Delta t) \in \{a_{\cdot,j}\}_\delta, N_j \le N\right) \le e^{2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)} \times \\
\times \prod_i P_{m(j-1)\Delta t}(\bar{m}_i(j\Delta t) \in \{a_{i,j}\}_\delta, N_{i,j} \le N) \qquad (3.83)
\end{aligned}
$$

and similarly for the lower bound. Last, it is easy to see that given an initial condition $m_i((j-1)\Delta t) \in \{a_{i,j-1}\}_\delta$, for every element of the set $\{\bar{m}_i(j\Delta t) \in \{a_{i,j}\}_\delta, N_{i,j} \le N\}$ corresponds only one element of $B^\delta_{i,j-1}(a)$, hence the right hand side of (3.83) equals that of (3.78), which concludes the proof of the lemma. $\square$

Remark 3.13. Note that if, instead of the definition (3.58) for the coarse potential, we used an alternative one which is also more common in the literature, e.g. see [60] formula (4.2.5.2), namely

$$
\bar{J}_\gamma(i,i') := \frac{1}{|I|^2}\int_{I_i\times I_{i'}} J_\gamma(r,r')dr\,dr', \quad i,i' \in \mathcal{I}, \qquad (3.84)
$$

then the estimate (3.80) would be simpler and equal to $c\delta$.

The next task is the asymptotic analysis of (3.76). In the lemma below we compute the cost functional for the Poisson process.

Lemma 3.14. *Given a profile $a \equiv \{a_{i,j}\}_{i,j} \in \bar{\Omega}_\gamma$, let $\nu^i = \mathbb{P}_{\gamma^{-1}|I|\bar{c}_+(i,a)} \times \mathbb{P}_{\gamma^{-1}|I|\bar{c}_-(i,a)}$ be the law of two independent Poisson processes with intensities $\gamma^{-1}|I|\bar{c}_+(i,a)$ and $\gamma^{-1}|I|\bar{c}_-(i,a)$, respectively. Then, for $d_{i,j-1} = \frac{a_{i,j}-a_{i,j-1}}{\Delta t}$ and $B^\delta_{i,j-1}(a)$ as in* (3.77)*, with some $\delta > 0$ small, e.g. $\delta = \Delta t\,\eta_0$, with η_0 as in* (3.30)*, we have:*

$$
\left|\frac{1}{\gamma^{-1}|I|}\ln\,\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a)) - \Delta t f(\hat{x}^\pm_{i,j-1};a)\right| \le \left(\frac{\delta}{\Delta t}\right)^{\frac{1-\alpha}{2}}\Delta t, \qquad (3.85)
$$

for $\alpha > 0$ small and where

$$
f(\hat{x}^\pm_{i,j-1};a) := h\left(\hat{x}^+_{i,j-1}\,|\,\bar{c}_+(i,a)\right) + h\left(\hat{x}^-_{i,j-1}\,|\,\bar{c}_-(i,a)\right). \qquad (3.86)
$$

Furthermore,

$$
h(z|\zeta) := z\ln\left(\frac{z}{\zeta}\right) - z + \zeta \qquad (3.87)
$$

and the optimal values $\hat{x}^\pm_{i,j-1}$ satisfy

$$
\hat{x}^+_{i,j-1}\hat{x}^-_{i,j-1} = \bar{c}_+(i,a)\bar{c}_-(i,a), \qquad 2(\hat{x}^-_{i,j-1} - \hat{x}^+_{i,j-1}) = d_{i,j-1}. \qquad (3.88)
$$

Remark 3.15. The error in (3.85) is negligible if we choose η_0 such that

$$\epsilon^{-3}\eta_0^{(1-\alpha)/2} \to 0, \quad \text{or } 3a - \frac{\lambda_0}{2}(1-\alpha) < 0. \tag{3.89}$$

Moreover, for later use, we also consider a Δt-dependent version of f in (3.86), namely:

$$f_{\Delta t}(\hat{x}^{\pm}_{i,j-1}; a) := h\left(\hat{x}^{+}_{i,j-1} \,|\, \Delta t\, \bar{c}_+(i,a)\right) + h\left(\hat{x}^{-}_{i,j-1} \,|\, \Delta t\, \bar{c}_-(i,a)\right). \tag{3.90}$$

Note that for the values $\hat{x}^{\pm}_{i,j-1}$ given in (3.88), the following is true:

$$f_{\Delta t}(\hat{x}^{\pm}_{i,j-1}; a) = f(\hat{x}^{\pm}_{i,j-1}; a) \cdot \Delta t.$$

The proof of the lemma will be given in Sect. 3.4.1. The next step is to show that the stochastic dynamics prefer to drive the system towards profiles $a \in \bar{\Omega}_\gamma$ which are away from the boundary values ± 1. We introduce the threshold

$$\delta' := \Delta t \cdot \eta_3, \quad \text{with } \eta_3 \equiv \eta_3(\gamma) := |\ln \gamma|^{-\lambda_3},\ \lambda_3 > 0, \tag{3.91}$$

where λ_3 will be determined in (3.120) and consider the class:

$$\bar{\Omega}^{\delta'}_\gamma := \{a \in \bar{\Omega}_\gamma : |a \pm 1| > \delta'\}. \tag{3.92}$$

In the following lemma we prove that given a profile $a \in \bar{\Omega}_\gamma$, we can construct a new profile $\tilde{a} \in \bar{\Omega}^{\delta'}_\gamma$ that the process m prefers to follow with higher or comparable probability.

Lemma 3.16. *Given any profile $a \equiv \{a_{i,j}\}_{i,j} \in \bar{\Omega}_\gamma$ and a threshold $\delta' := \Delta t \cdot \eta_3$ as defined in* (3.91) *where η_3 satisfies the following constraint*

$$3a - \lambda_3(1-\alpha) < 0, \tag{3.93}$$

$\forall \alpha > 0$ small, there exists a profile $\tilde{a} \in \bar{\Omega}^{\delta'}_\gamma$ (which can be constructed explicitly), such that $|1 \pm \tilde{a}| \geq \delta'$ and the following bound holds:

$$\frac{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))}{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(\tilde{a}))} \leq e^{\gamma^{-1}|I|\Delta t\, \eta_3^{1-\alpha}}.$$

Remark 3.17. Note that the error is negligible if we take $\gamma \ln()$ and consider all space-time coarse-grained boxes, i.e.,

$$\gamma \frac{\epsilon^{-3}}{|I|\Delta t} \gamma^{-1} |I| \Delta t\, \eta_3^{1-\alpha} = \epsilon^{-3}\eta_3^{1-\alpha} \to 0,$$

under the constraint (3.93).

The proof is given in Sect. 3.4.2. We summarize what we have done so far: by putting together the results of Lemmas 3.9, 3.11, 3.14 and 3.16 and considering the number of all time-space coarse cells, we have the following lower and upper bounds, for $\gamma > 0$ small enough and for some $c > 0$:

(**Lower Bound**) For a profile $a \equiv \{a_{i,j}\}_{i,j} \in \bar{\Omega}_\gamma^{\delta'}$ we have

$$P_{\sigma_{(j-1)\Delta t}}(\sigma_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j \le N) \ge \epsilon^{-2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)} e^{-c\epsilon^{-1}\gamma^{-1}\Delta t(\eta_0^{(1-\alpha)/2}+\eta_3^{(1-\alpha)})} \prod_{i\in\mathcal{I}} e^{-\gamma^{-1}|I|\Delta t f(\hat{x}^{\pm}_{i,j-1};a)}. \tag{3.94}$$

(**Upper Bound**) For a profile $a \equiv \{a_{i,j}\}_{i,j} \in \bar{\Omega}_\gamma$, there exists a profile $\tilde{a} \equiv \{\tilde{a}_{i,j}\}_{i,j} \in \bar{\Omega}_\gamma^{\delta'}$ such that

$$P_{\sigma_{(j-1)\Delta t}}(\sigma_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j \le N) \;\le\; e^{2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)} e^{c\epsilon^{-1}\gamma^{-1}\Delta t(\eta_0^{(1-\alpha)/2}+\eta_3^{(1-\alpha)})} \prod_{i\in\mathcal{I}} e^{-\gamma^{-1}|I|\Delta t f(\hat{x}^{\pm}_{i,j-1};\tilde{a})}. \tag{3.95}$$

Note that the error is negligible under the requirements in the corresponding lemmas.

3.3.6 Bad time intervals

Going back to (3.56) and the discussion below, for the terms in $\{a\}_\delta \cap D^c_{\bar{k}}$ with $j \notin \{j_1^*, \dots, j_k^*\}$ we use the formula derived in the previous section. On the other hand, for the terms with $j \in \{j_1^*, \dots, j_k^*\}$ we consider upper and lower bounds by replacing the rates by the corresponding constant ones c_m and c_M as in (3.8). Hence, for the case of the upper bound (and similarly for the lower bound), we construct a new process $\tilde{\sigma}$ which is a Markov Process with infinitesimal transition probability $\tilde{P}$ given by:

$$\tilde{P}(\tilde{\sigma}_{n+1} = \tilde{\sigma}',\, t \le t_{n+1} < t+dt \,|\, \tilde{\sigma}_n = \tilde{\sigma},\, t_n = s) = c_M \sum_{x\in\mathcal{S}_\gamma} \mathbf{1}_{\tilde{\sigma}'=\tilde{\sigma}^x} e^{-c_M|\mathcal{S}_\gamma|(t-s)} \mathbf{1}_{\{t>s\}} dt. \tag{3.96}$$

In the new process we have replaced the rates by constant ones in such a way to get an upper bound. It is easy to check that

$$P_{\sigma_{(j-1)\Delta t}}(\sigma_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j > N) \;\le\; e^{-(c_m-c_M)2\gamma^{-1}\epsilon^{-1}L\Delta t} \times \times \tilde{P}_{\sigma_{(j-1)\Delta t}}(\tilde{\sigma}_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j > N) \tag{3.97}$$

and

$$P_{\sigma_{(j-1)\Delta t}}(\sigma_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j > N) \;\ge\; e^{-(c_M-c_m)2\gamma^{-1}\epsilon^{-1}L\Delta t} \times \times \tilde{P}_{\sigma_{(j-1)\Delta t}}(\tilde{\sigma}_{j\Delta t} \in \{a_{\cdot,j}\}_\delta, N_j > N), \tag{3.98}$$

where $\tilde{P}$ is the law of the new process $\{\tilde{\sigma}_t\}_{t\geq 0}$. To compute the upper and lower bounds for the new process we proceed as before and consider the corresponding mean field process $\{\tilde{m}_i(t)\}_{i\in\mathcal{I},\, t\geq 0}$ with rates given by

$$c_+(i,\tilde{m}) = \bar{k}_+^{i,j-1}(a_{i,j-1})c_M \qquad \text{and} \quad c_-(i,\tilde{m}) = \bar{k}_-^{i,j-1}(a_{i,j-1})c_m.$$

By defining the Poisson representation of the process in a similar fashion as in Sect. 3.3.4 we obtain similar upper (g_1) and lower (g_2) bounds as in (3.94) and (3.95), respectively, where instead of f we have

$$g_1(\hat{z}^{\pm}_{i,j-1};a) = h\left(\hat{z}^{-}_{i,j-1}\,|\,\gamma^{-1}|I|k_-^{i,j-1}(a_{i,j-1})c_M\right) + h(\hat{z}^{+}_{i,j-1}\,|\,\gamma^{-1}|I|k_+^{i,j-1}(a_{i,j-1})c_M) \tag{3.99}$$

$$g_2(\hat{z}^{\pm}_{i,j-1};a) = h\left(\hat{z}^{-}_{i,j-1}\,|\,\gamma^{-1}|I|k_-^{i,j-1}(a_{i,j-1})c_m\right) + h(\hat{z}^{+}_{i,j-1}\,|\,\gamma^{-1}|I|k_+^{i,j-1}(a_{i,j-1})c_m\right). \tag{3.100}$$

Here $\hat{z}^{\pm}$ are computed following the Sect. 3.4.1. Note that we also have a rough lower bound: $g_{1,2}(\hat{z}^{\pm};a) \geq -c_b$ where c_b is a positive constant number since $h \geq 0$.

Now we have all the ingredients to derive the discrete version of the cost functional in the space $\Lambda_\epsilon \times \mathcal{T}_\epsilon$.

3.3.7 Derivation of the cost functional

We recall from Definition 3.8 the space $\mathrm{PC}_{|I|}\mathrm{Aff}_{\Delta t}(\Lambda_\epsilon \times \mathcal{T}_\epsilon)$ of all functions

$$\phi_a(x,t) := \sum_i \mathbf{1}_{I_i}(x) \sum_j \mathbf{1}_{[(j-1)\Delta t, j\Delta t)}(t) \left[\frac{a_{i,j} - a_{i,j-1}}{\Delta t} t + j\cdot a_{i,j-1} - (j-1)\cdot a_{i,j}\right], \tag{3.101}$$

which are linear interpolation between the values $a(x,(j-1)\Delta t)$ and $a(x,j\Delta t))$ and piece-wise constant in space. We also consider another function which agrees with its derivative in each open interval:

$$\psi_a(x,t) = \sum_{i,j} \frac{a_{i,j} - a_{i,j-1}}{\Delta t} \mathbf{1}_{I_i}(x)\mathbf{1}_{[(j-1)\Delta t, j\Delta t)}(t). \tag{3.102}$$

We also recall that $\{k^*, j_1^*, \ldots, j_{k^*}^*\} = \mathrm{argmax}_{\{k,j_1,\ldots,j_k\}} P(\{a\}_\delta \cap D^{(k)}_{j_1,\ldots,j_k})$ and for simplicity we call $J^* := \{j_1^*, \ldots, j_{k^*}^*\}$. Then, for $a \in \bar{\Omega}_\gamma$, from (3.56), (3.95), (3.97) and

(3.100) we get

$$
\begin{aligned}
P(\{a\}_\delta\cap D_{\bar k}^c) &\le \bar k \binom{\frac{\epsilon^{-2}T}{\Delta t}}{k} \max_{k,j_1,\dots,j_k} P(\{a\}_\delta\cap D^{(k)}_{j_1,\dots,j_k})\\
&\le e^{\frac{\epsilon^{-2}}{\Delta t}2c\beta L\epsilon^{-1}\gamma^{-1}\frac{1}{\eta_1}\Delta t(C^*(\gamma)+\delta)}\times e^{c\epsilon^{-3}(\eta_0^{(1-\alpha)/2}+\eta_3^{(1-\alpha)})}\times\\
&\quad\times e^{-\bar k(c_m-c_M)2\gamma^{-1}\epsilon^{-1}L\Delta t}\times\prod_{j\notin J^*}\prod_i e^{-\gamma^{-1}|I|\Delta t\, f(\hat x^{\pm}_{i,j};\tilde a)+o_\gamma(1)}\times\\
&\quad\times\prod_{j\in J^*}\prod_i e^{-\gamma^{-1}|I|\Delta t\, g_1(\hat z^{\pm}_{i,j};\tilde a)+o_\gamma(1)},
\end{aligned}\tag{3.103}
$$

for some $\tilde a\in\bar\Omega_\gamma^{\delta'}$ and $\bar k$ as in (3.54). A similar lower bound is obtained following the same reasoning. To have a negligible error in (3.103) we need the constraints (3.63), (3.89), (3.93) and

$$
\gamma\bar k\gamma^{-1}\epsilon^{-1}\Delta t\to 0 \quad\text{or}\quad \eta_1<<\eta_2,\ \text{ i.e., }\ \lambda_1>\lambda_2,\tag{3.104}
$$

which is true from the choice made in (3.55).

The next step is to replace f by the density H of the cost functional:

Lemma 3.18. *For every $a\in\bar\Omega_\gamma^{\delta'}$, with δ' as in* (3.91)*, ϕ_a and ψ_a as in* (3.101) *and* (3.102)*, there is a constant $C_\gamma\to 0$ as $\gamma\to 0$ such that*

$$
\|F(\hat x^{\pm};a)-H(\phi_a,\psi_a)\|_{L^1(\Lambda_\epsilon\times\mathcal T_\epsilon)}\le C_\gamma.\tag{3.105}
$$

Both $F(\hat x^{\pm};a)$ and $H(\phi_a,\psi_a)$ are functions in $\Lambda_\epsilon\times\mathcal T_\epsilon$ given by:

$$
F(\hat x^{\pm};a)(x,t):=\sum_{i\in\mathcal I}\mathbf 1_{I_i}(x)\sum_{j\in\mathcal J}\mathbf 1_{[(j-1)\Delta t,j\Delta t)}(t)f(\hat x^{\pm}_{i,j};a),\tag{3.106}
$$

with $f(\hat x^{\pm}_{i,j};a)$ as given in (3.86) *and*

$$
\begin{aligned}
H(\phi_a,\psi_a):=&\frac{\psi_a}{2}\left[\ln\frac{\psi_a+\sqrt{(1-\phi_a^2)(1-\tanh^2(\beta J*\phi_a))+\psi_a^2}}{(1-\phi_a)\sqrt{1-\tanh^2(\beta J*\phi_a)}}-\beta J*\phi_a\right]\\
&+\frac12\Big[1-\phi_a\tanh(\beta J*\phi_a)-\\
&\quad-\sqrt{(1-\phi_a^2)(1-\tanh^2(\beta J*\phi_a))+\psi_a^2}\Big],
\end{aligned}\tag{3.107}
$$

where the x,t dependence is hidden in ϕ_a and ψ_a.

Proof. We first estimate the difference between $\hat x^{+}_{i,j}$ as in (3.131) and $y\equiv y(\phi_a,\psi_a)$ with

$$
y(\phi_a,\psi_a)=-\frac{\psi_a}{4}+\sqrt{\frac{\psi_a^2}{16}+c_+(\phi_a)c_-(\phi_a)}.\tag{3.108}
$$

The rates $c_\pm(\phi_a)$ are defined analogously to $\bar{c}_\pm(i,a)$ in (3.74) where instead of $a_{j-1}(x)$ we have ϕ_a, that is,

$$c_\pm(\phi_a) := \frac{1 \pm \phi_a}{2} F_\mp (J * \phi_a) .$$

By comparing to the rates $\bar{c}_\pm(i,a)$ we obtain that for $x \in I_i$ and $t \in [(j-1)\Delta t, j\Delta t)$:

$$|\hat{x}^+_{i,j} - y(\phi_a,\psi_a)(x,t)| \le c|\psi_a(x,t)\Delta t|^{1/2}, \tag{3.109}$$

for some $c > 0$. Moreover the following identities are satisfied by the above rates,

$$F_- (J * \phi_a) \cdot F_+ (J * \phi_a) = \frac{1}{(e^{\beta J * \phi_a} + e^{-\beta J * \phi_a})^2} = \frac{1}{4}(1 - \tanh^2(\beta J * \phi_a))$$

and

$$c_+(\phi_a) + c_-(\phi_a) = \frac{1}{2}[1 - \phi_a \tanh(\beta J * \phi_a)].$$

From these and after some straightforward cancellations, we rewrite the function $H(\phi_a,\psi_a)$ in (3.107) as follows:

$$H(\phi_a,\psi_a) = h\left(y(\phi_a,\psi_a) \,|\, c_+(\phi_a)\right) + h(y(\phi_a,\psi_a) + \frac{\psi_a}{2} \,|\, c_-(\phi_a)),$$

where h is defined in (3.87). Notice the similarity with $f(\hat{x}^\pm_{i,j};a)$, where ϕ_a and $c_\pm(\phi_a)$ have replaced a and $\bar{c}_\pm(i,a)$, respectively.

Then, for the difference $|f(\hat{x}^\pm_{i,j};a) - H(\phi_a,\psi_a)|$, it suffices to estimate the following as the other terms can be treated in a similar fashion:

$$\left| \hat{x}^+_{i,j} \ln \frac{\hat{x}^+_{i,j}}{1+a} - y \ln \frac{y}{1+\phi_a} \right| \le |\hat{x}^+_{i,j} - y|^{1-\alpha} + |\hat{x}^+_{i,j}| \cdot |\ln \frac{1+a}{1+\phi_a}| + |\hat{x}^+_{i,j} - y| \cdot |\ln(1+\phi_a)|.$$

The first term is given in (3.109) so we require that

$$\epsilon^{-3}|\dot{\phi}_a \Delta t|^{(1-\alpha)/2} \to 0, \qquad \text{as} \quad \gamma \to 0. \tag{3.110}$$

Note that if all allowed spin-flips occur on the same space coarse-grained box we have the bound

$$|\dot{\phi}_a| \le \frac{N}{\gamma^{-1}|I|} \le \frac{\epsilon^{-1}}{\eta_1 \cdot |I|}, \tag{3.111}$$

where N were chosen in (3.50). Thus, requirement (3.110) is easily satisfied since $\Delta t = \gamma^c$.

The main difficulty is in the second term since, in some regimes, $|\hat{x}^+_{i,j}|$ may be large and at the same time $1+\phi_a$ small. This occurs when the given profile a (and subsequently also ϕ_a) is very close to the boundary value -1 (recall the lower bound $1+\phi_a \geq \Delta t \cdot \eta_3$ from (3.92)) with a negative derivative which can also be large in absolute value, given by (4.53). Due to the symmetry of the problem the same holds for the case of a profile going up and being close to the upper boundary $+1$ in which case the “bad" term is $|\hat{x}^-_{i,j}| \cdot |\ln \frac{1-a}{1-\phi_a}|$. More precisely, in (3.129), if $\frac{d_{i,j}}{4} < -\sqrt{\frac{B(a,\Delta t)}{(\Delta t)^2}} < 0$, then $|\hat{x}^+_{i,j}| \lesssim |\frac{d_{i,j}}{4}| \lesssim \frac{\epsilon^{-1}}{\eta_1 \cdot |I|}$. We fix a threshold

$$\eta_4 \equiv \eta_4(\gamma) := |\ln \gamma|^{-\lambda_4}, \;\; \lambda_4 > 0, \tag{3.112}$$

such that $\eta_4 >> \Delta t$ and we split the integral $\int |\hat{x}^+_{i,j}| \cdot |\ln \frac{1+a}{1+\phi_a}|\, dx\, dt$ into the set $\{1+\phi_a > \frac{\Delta t}{\eta_4}\}$ and its complement. For the first we have that

$$\frac{1+a}{1+\phi_a} = 1 + \frac{a-\phi_a}{1+\phi_a}, \;\; \text{where} \;\; \left|\frac{a-\phi_a}{1+\phi_a}\right| \leq \frac{|\psi_a| \cdot \Delta t}{\frac{\Delta t}{\eta_4}} \leq \frac{\eta_4 \cdot \epsilon^{-1}}{\eta_1 \cdot |I|}$$

and we choose

$$\Delta t << \eta_4 << \epsilon \cdot \eta_1 \cdot |I|. \tag{3.113}$$

Under this condition we obtain that

$$\int_{\{1+\phi_a > \frac{\Delta t}{\eta_4}\}} |\hat{x}^+_{i,j}| \cdot \left|\ln \frac{1+a}{1+\phi_a}\right| \, dx\, dt \leq \epsilon^{-4} \frac{\eta_4 \cdot \epsilon^{-1}}{\eta_1^2 \cdot |I|^2}. \tag{3.114}$$

This is vanishing provided that

$$\eta_4 << \eta_1^2 \cdot |I|^2 \cdot \epsilon^4, \;\; \text{i.e.,} \;\; \lambda_4 > 2\lambda_1 + 2b + 4a, \tag{3.115}$$

which also covers the previous requirement (3.113).

In the complement, recalling the properties (3.25) of the functional, we exploit the fact that $\psi_a \ln(1+\phi_a) \in L^1(\Lambda_\epsilon \times \mathcal{T}_\epsilon)$ for $\psi_a = \dot{\phi}_a$. Indeed, we have that:

$$P > \int_{\{1+\phi_a \leq \frac{\Delta t}{\eta_4}\}} |\psi_a| \cdot |\ln(1+\phi_a)| \, dx\, dt > \ln \Delta t \int_{\{1+\phi_a \leq \frac{\Delta t}{\eta_4}\}} |\psi_a|. \tag{3.116}$$

On the other hand, we also have that $\frac{1+a}{1+\phi_a} > 1$ which implies that

$$\left|\ln \frac{1+a}{1+\phi_a}\right| \leq \left|\frac{1+a}{1+\phi_a} - 1\right| = \left|\frac{a-\phi_a}{1+\phi_a}\right| \leq \frac{\epsilon^{-1}}{\eta_1 |I| \eta_3}, \tag{3.117}$$

from (4.53) and the fact that $|1+\phi_a| \geq \Delta t\, \eta_3$. From (3.116) and (3.117) we obtain:

$$\int_{\{1+\phi_a \leq \frac{\Delta t}{\eta_4}\}} |\hat{x}^+_{i,j}| \cdot \left|\ln \frac{1+a}{1+\phi_a}\right| \, dx\, dt \leq \frac{\epsilon^{-1}}{\eta_1 |I| \eta_3} \cdot \frac{P}{\ln \Delta t}, \tag{3.118}$$

which is vanishing under the requirement that

$$|\ln \Delta t|^{-1} << \eta_1 |I| \eta_3 \cdot \epsilon, \text{ i.e., } 1 > \lambda_1 + b + \lambda_3 + a. \tag{3.119}$$

It is easy to check that the requirements (3.63) for η_1, (3.93) for η_3 and (3.119) for both, can be simultaneously satisfied, e.g. by choosing λ_1 and λ_3 such that

$$1 > 2\lambda_1 + \frac{4}{3}\lambda_3(1-\alpha). \tag{3.120}$$

Then the other parameters can be chosen as follows: η_0 from requirement (3.89), η_2 from (3.104) and η_4 from (3.115). The parameters a and b, for ϵ and $|I|$ respectively, have more freedom, but within the limits of the above constraints. Finally, the error C_γ in (3.105) is given by the right hand sides of (3.114) and (3.118) which are vanishing as $\gamma \to 0$. □

Putting together good and bad time intervals from (3.85) and (3.99)-(3.100), we obtain the following bound for the last two factors of (3.103):

$$\exp\left\{ -\gamma^{-1} \left(\sum_{i \in \mathcal{I}} \Big(\sum_{j \in J^*} g_{1,2}(\hat{z}^{\pm}_{i,j}; a) + \sum_{j \notin J^*} f(\hat{x}^{\pm}_{i,j}; a) \Big) |I| \Delta t \right) \right\}, \tag{3.121}$$

since both f and $g_{1,2}$ are integrable functions in $\Lambda_\epsilon \times \mathcal{T}_\epsilon$ and $|J^*|/(\epsilon^{-2}/\Delta t)$ is negligible. Using again Lemma 3.18 we have that (3.121) equals $\int_{\Lambda_\epsilon \times \mathcal{T}_\epsilon} H(\phi_a, \psi_a) dx\, dt$ plus vanishing error as $\gamma \to 0$. We conclude the last step of the strategy (3.45) by restricting to the class of smoother functions:

Lemma 3.19. *Given a closed set $C \subset D(\mathbb{R}_+, \{-1,+1\}^{\mathcal{S}_\gamma})$, for some $\delta, \delta' > 0$ we denote by $\bar{\Omega}^{\delta'}_{\gamma,\delta}(C)$ the set of profiles in $\bar{\Omega}_{\gamma,\delta}(C)$ defined in* (3.32)*, with the extra property that $|a \pm 1| > \delta'$. Then, for such a profile $a \in \bar{\Omega}^{\delta'}_{\gamma,\delta}(C)$ and δ, δ' chosen as before, we have that*

$$\inf_{a \in \bar{\Omega}^{\delta'}_{\gamma,\delta}(C)} \int_{\Lambda_\epsilon \times \mathcal{T}_\epsilon} H(\phi_a, \psi_a) dx\, dt \geq \inf_{\phi \in \mathcal{U}_\delta(C)} I_{\Lambda_\epsilon \times \mathcal{T}_\epsilon}(\phi) + C_\gamma, \tag{3.122}$$

with the same C_γ as in (3.105)*.*

Proof. Mollified versions of (ϕ_a, ψ_a) are elements in $\mathcal{U}_\delta(C)$ to which we can restrict ourselves by obtaining a lower bound. Furthermore, mollified functions are close in L^1 to the original ones. The same is true for their images under integrable functions such as the ones in $H(\phi_a, \dot{\phi}_a)$. Hence, we can approximate $H(\phi_a, \dot{\phi}_a)$ by H evaluated at mollified versions of ϕ_a with a negligible error which is similar to the one in Lemma 3.18. This is a standard calculation and details are omitted. □

3.4 Properties of the Poisson process

In this section we obtain an asymptotic formula for the logarithm of the Poisson distribution. Before proceeding with the proof of Lemma 3.14, we establish some notation. For every $i \in \mathcal{I}$ and $j \in \mathcal{J}$, we define the random variables

$$X^{i,j-1} := \frac{N^+_{i,j-1}}{\gamma^{-1}|I|} \tag{3.123}$$

and

$$K^{i,j} := \frac{2\big(N^-_{i,j-1} - N^+_{i,j-1}\big)}{\gamma^{-1}|I|}. \tag{3.124}$$

Given $a \in \bar{\Omega}_\gamma$, we denote by $R^\delta_{i,j}(a)$ the range of the values that the pair of random variables $(X^{i,j-1}, K^{i,j})$ can take. This is determined by the set $\{|K^{i,j} - d_{i,j-1}\Delta t| < \delta\}$ for $K^{i,j}$ and by the set $[m^\delta_{i,j}(K^{i,j}), M^\delta_{i,j}(a)]$ for $N^+_{i,j-1}$. In the latter, we have defined

$$m^\delta_{i,j}(K^{i,j}) := \max\left\{0, -\gamma^{-1}|I|K^{i,j}\right\}, \tag{3.125}$$

$$M^\delta_{i,j}(a) := \gamma^{-1}|I| \cdot \left(\min\left\{\bar{k}^{i,j-1}_+(a), \bar{k}^{i,j-1}_-(a) - K^{i,j}\right\} - \delta\right), \tag{3.126}$$

for the lower and upper limits (respectively) of the potential values of $X^{i,j-1}$, given $d_{i,j-1}$ as in Lemma 3.11 and $\bar{k}^{i,j-1}_\pm(a)$ in (3.75). Note that in $M^\delta_{i,j}(a)$ the minimum is over the number of pluses at time $(j-1)\Delta t$ and the number of minuses at the next time $j\Delta t$, as the number of pluses that become minuses cannot exceed neither of them. By (3.76) we have:

$$\begin{aligned}
&\nu^i_{m_i((j-1)\Delta t)}\big(B^\delta_{i,j-1}(a)\big) = \\
&= \sum_{(n^-_{i,j-1}, n^+_{i,j-1}) \in B^\delta_{i,j}} \mathbb{P}_{\gamma^{-1}|I|\bar{c}_-(i,a)}\big(N^-_{i,j-1} = n^-_{i,j-1}\big)\, \mathbb{P}_{\gamma^{-1}|I|\bar{c}_+(i,a)}\big(N^+_{i,j-1} = n^+_{i,j-1}\big) \\
&= \sum_{(n^+_{i,j-1}, k^{i,j}) \in R^\delta_{i,j}(a)} \mathbb{P}_{\gamma^{-1}|I|\bar{c}_-(i,a)}\big(N^-_{i,j-1} = n^+_{i,j-1} + \frac{\gamma^{-1}|I|k^{i,j}}{2}\big) \times \\
&\qquad \times \mathbb{P}_{\gamma^{-1}|I|\bar{c}_+(i,a)}\big(N^+_{i,j-1} = n^+_{i,j-1}\big).
\end{aligned} \tag{3.127}$$

For $n^+_{i,j-1}$ and $n^+_{i,j-1} + \gamma^{-1}|I|k^{i,j}$ large enough, we apply Stirling's formula to (3.127) and using (3.76) we obtain the following expression:

$$\sum_{(x^+_{i,j-1}, k^{i,j}) \in \gamma^{-1}|I|R^\delta_{i,j}(a)} \exp\left(-\gamma^{-1}|I| f_{\Delta t}(x^\pm_{i,j-1}; a) + o_\gamma(1)\right), \tag{3.128}$$

where $f_{\Delta t}(x^\pm_{i,j-1}; a)$ is given in (3.90) and $x^\pm_{i,j-1}$ represents the number of occurrence of the random times $N^\pm_{i,j-1}$ divided by $\gamma^{-1}|I|$. Recall also that $x^-_{i,j-1} = x^+_{i,j-1} + \gamma^{-1}|I|k^{i,j}$.

Moreover, note that in the latter sum, $k^{i,j}$ denotes a rescaled number by $\gamma^{-1}|I|$ while in the sum in (3.127) it is not rescaled.

3.4.1 Asymptotics of the Poisson process, proof of Lemma 3.14

We give the asymptotic analysis of the Poisson Process.

Proof of Lemma 3.14. We optimize the exponent of (3.128) with respect to $x^{+}_{i,j-1} \in \gamma^{-1}|I|R^{\delta}_{i,j}(a)$ (viewing $k^{i,j}$ as a parameter) and using the fact that $x^{-}_{i,j-1} = x^{+}_{i,j-1} + \gamma^{-1}|I|k^{i,j}$. The optimal value is given by

$$x^{+,\text{opt}}_{i,j-1} = -A(k^{i,j}) + \sqrt{A(k^{i,j})^2 + B(a,\Delta t)} \geq 0, \tag{3.129}$$

where

$$A(k^{i,j}) = \frac{k^{i,j}}{4} \quad \text{and} \quad B(a,\Delta t) = \bar{c}_{+}(i,a)\bar{c}_{-}(i,a)(\Delta t)^2.$$

Calling

$$\bar{A}(a,\Delta t) := \frac{d_{i,j-1}}{4}\Delta t,$$

we define

$$\begin{aligned}
\bar{x}^{+}_{i,j-1} &:= -\bar{A}(a,\Delta t) + \sqrt{\bar{A}(a,\Delta t)^2 + B(a,\Delta t)} \\
&= \Delta t \left(-\frac{d_{i,j-1}}{4} + \sqrt{\frac{d^2_{i,j-1}}{16} + \bar{c}_{+}(i,a)\bar{c}_{-}(i,a)} \right) \\
&=: \Delta t\, \bar{y}_{i,j-1}(a).
\end{aligned} \tag{3.130}$$

By using the second property of the set $R^{\delta}_{i,j}(a)$, namely that $|k^{i,j} - d_{i,j-1}\Delta t| < \delta$ and comparing (3.129) and (3.130) we have that:

$$\left|\frac{x^{+,\text{opt}}_{i,j-1}}{\Delta t} - \bar{y}_{i,j-1}(a)\right| \leq \frac{1}{2}\frac{\delta}{\Delta t} + \left(\frac{\delta}{\Delta t}\right)^{1/2},$$

which implies that

$$\left|\frac{x^{+,\text{opt}}_{i,j-1}}{\Delta t} \ln \frac{x^{+,\text{opt}}_{i,j-1}}{\Delta t\, \bar{c}_{+}(i,a)} - \bar{y}_{i,j-1}(a) \ln \frac{\bar{y}_{i,j-1}(a)}{\bar{c}_{+}(i,a)}\right| \leq$$

$$\leq \left|\frac{x^{+,\text{opt}}_{i,j-1}}{\Delta t} - \bar{y}_{i,j-1}(a)\right|^{1-\alpha} + \left|\frac{x^{+,\text{opt}}_{i,j-1}}{\Delta t} - \bar{y}_{i,j-1}(a)\right| \cdot |\ln c_{+}(i,a)| \leq \left(\frac{\delta}{\Delta t}\right)^{\frac{1-\alpha}{2}}.$$

Thus,

$$\left|h\left(x^{+,\text{opt}}_{i,j-1} \,|\, \Delta t\, \bar{c}_{+}(i,a)\right) - \Delta t\, h\left(\bar{y} \,|\, c_{+}(i,a)\right)\right| \leq \left(\frac{\delta}{\Delta t}\right)^{\frac{1-\alpha}{2}} \Delta t.$$

We treat the term $h\left(x^{+,\mathrm{opt}}_{i,j-1}+\frac{k^{i,j}}{2}\mid \bar c_-(i,a)\right)$ similarly. Thus, the optimal values are

$$\hat x^+_{i,j-1} := \bar y_{i,j-1}(a) \quad \text{and} \quad \hat x^-_{i,j-1} := \frac{d_{i,j-1}}{2}+\bar y_{i,j-1}(a). \tag{3.131}$$

Thus, we substitute them in (3.128) and since the cardinality of the sum is negligible after we take $\gamma\ln()$, we conclude the proof of the lemma. □

3.4.2 Move profiles away from ± 1, proof of Lemma 3.16

We show that the stochastic dynamics drive the magnetization profile away from the boundaries ± 1.

Proof of Lemma 3.16. Whenever the profile a enters the safety region $|1\pm a|\leq \delta'$ we move it away from it by δ'. We define a new profile $\tilde a$ as follows:

$$\tilde a_{i,j} := (a_{i,j}-\delta')\mathbf{1}_{\{a_{i,j}>1-\delta'\}}+a_{i,j}\mathbf{1}_{\{-1+\delta'\leq a_{i,j}\leq 1-\delta'\}}+(a_{i,j}+\delta')\mathbf{1}_{\{a_{i,j}<-1+\delta'\}}, \tag{3.132}$$

with δ' as in (3.91) under the constraint (3.93) and by choosing it to be a multiple of Δ we have that $\tilde a_{i,j}\in\bar\Omega_\gamma$. Next, we consider the case when the fixed configuration a is close to the $+1$ boundary, with the other case being similar due to the symmetry of the problem.

It is more convenient to slightly change the notation for $f_{\Delta t}(x^\pm_{i,j-1};a)$ making explicit the dependence on $k^{i,j}$, i.e., writing $f_{\Delta t}((x^\pm_{i,j-1},k^{i,j});a)\equiv f_{\Delta t}(x^\pm_{i,j-1};a)$. Then, the strategy goes as follows: we seek an injective map ι in such a way that the following two inequalities are true:

$$\begin{aligned}
\frac{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))}{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(\tilde a))} &= \frac{\sum_{(x^+_{i,j-1},k^{i,j})\in R^\delta_{i,j}(a)}\frac{e^{-\gamma^{-1}|I|f_{\Delta t}((x^+_{i,j-1},k^{i,j});a)}}{e^{-\gamma^{-1}|I|f_{\Delta t}(\iota(x^+_{i,j-1},k^{i,j});\tilde a)}}e^{-\gamma^{-1}|I|f_{\Delta t}(\iota(x^+_{i,j-1},k^{i,j});\tilde a)}}{\sum_{(\tilde x^+_{i,j-1},\tilde k^{i,j})\in R^\delta_{i,j}(\tilde a)}e^{-f_{\Delta t}(\tilde x^+_{i,j-1},\tilde k^{i,j};\tilde a)}}\\
&\leq e^{M(\gamma)}\frac{\sum_{(x^+_{i,j-1},k^{i,j})\in R^\delta_{i,j}(a)}e^{-\gamma^{-1}|I|f_{\Delta t}(\iota(x^+_{i,j-1},k^{i,j});a)}}{\sum_{(\tilde x^+_{i,j-1},\tilde k^{i,j})\in R^\delta_{i,j}(\tilde a)}e^{-\gamma^{-1}|I|f_{\Delta t}((\tilde x^+_{i,j-1},\tilde k^{i,j});\tilde a)}}\leq\\
&\leq e^{M(\gamma)},
\end{aligned}\tag{3.133}$$

for some $M(\gamma)$ to be estimated.

Definition of the injective map ι. We have three cases: suppose that the profile a is close to the $+1$ boundary at time $(j-1)\Delta t$, $j\Delta t$ or both. For every $(x^+_{i,j-1},k^{i,j})\in R^\delta_{i,j}(a)$ we choose a pair $(\tilde x^+_{i,j-1},\tilde k^{i,j}):=\iota(x^+_{i,j-1},k^{i,j})\in R^\delta_{i,j}(\tilde a)$ by replacing $d_{i,j-1}$ by

$$\tilde d_{i,j-1}=\frac{\tilde a_{i,j}-\tilde a_{i,j-1}}{\Delta t},$$

with $\tilde{a}_{i,j-1} = a_{i,j-1} - \delta'$ or $\tilde{a}_{i,j} = a_{i,j} - \delta'$, respectively. Then, for the first inequality of (3.133), the difference

$$-f_{\Delta t}(x^+_{i,j-1}, k^{i,j}; a) + f_{\Delta t}(\iota(x^+_{i,j-1}, k^{i,j}); \tilde{a}) =$$

$$= -x^+ \ln \frac{x^+}{c_+(i,a)\Delta t} + \tilde{x}^+ \ln \frac{\tilde{x}^+}{c_+(i,\tilde{a})\Delta t} + 2(x^+ - \tilde{x}^+) + \Delta t \left(c_+(i,\tilde{a}) - c_+(i,a)\right) -$$

$$- (x^+ + \frac{k}{2}) \ln \frac{x^+ + \frac{k}{2}}{c_-(i,a)\Delta t} + (\tilde{x}^+ + \frac{k}{2}) \ln \frac{\tilde{x}^+ + \frac{\tilde{k}}{2}}{c_-(i,\tilde{a})\Delta t} + \frac{1}{2}(k - \tilde{k}) + \Delta t \left(c_-(i,\tilde{a}) - c_-(i,a)\right), \tag{3.134}$$

can be estimated using the following inequalities:

$$-x^+ \ln \frac{x^+}{c_+(i,a)\Delta t} + \tilde{x}^+ \ln \frac{\tilde{x}^+}{c_+(i,\tilde{a})\Delta t} \leq \Delta t \left| \frac{x^+ - \tilde{x}^+}{\Delta t} \right|^{1-\alpha} + \tilde{x}^+ \ln \frac{c_+(i,a)}{c_+(i,\tilde{a})} + \\ +(x^+ - \tilde{x}^+) \ln c_+(i,a) \tag{3.135}$$

and

$$-(x^+ + \frac{k}{2}) \ln \frac{x^+ + \frac{k}{2}}{c_-(i,a)} + (\tilde{x}^+ + \frac{\tilde{k}}{2}) \ln \frac{\tilde{x}^+ + \frac{\tilde{k}}{2}}{c_-(i,\tilde{a})} \leq \Delta t \cdot \left| \frac{x^+ - \tilde{x}^+ + \frac{1}{2}(k - \tilde{k})}{\Delta t} \right|^{1-\alpha} +$$

$$+ \left(\tilde{x}^+ + \frac{\tilde{k}}{2}\right) \cdot \ln \frac{c_-(i,a)}{c_-(i,\tilde{a})} + \left(x^+ - \tilde{x}^+ + \frac{1}{2}(k - \tilde{k})\right) \cdot \ln c_-(i,a), \tag{3.136}$$

where $\alpha \in (0,1)$. Note that for notational simplicity, in some variables we removed the indices i, j denoting dependence on the box.

For the second inequality of (3.133) in all three cases we show that $|R^\delta_{i,j}(a)| < |R^\delta_{i,j}(\tilde{a})|$.

<u>*Case 1:*</u> **The profile a enters the safety zone.** When the profile a enters the safety zone, the new profile $\tilde{a}$ is defined as $\tilde{a}_{i,j-1} := a_{i,j-1}$ and $\tilde{a}_{i,j} := a_{i,j} - \delta'$. We choose $\tilde{x}^+ := x^+$ and $\tilde{k} := k - \frac{\delta'}{2}$, i.e., we keep the same number of plus jumps and we reduce the number of minus jumps. We also have that

$$\tilde{d} = d - \frac{\delta'}{\Delta t} \quad \text{and} \quad c_\pm(i,a) = c_\pm(i,\tilde{a}).$$

So in (3.134) there is no contribution to the error from the comparison of x and $\tilde{x}$ and we only estimate the terms that correspond to the number of minus, as in (3.136). Moreover, the last term in the r.h.s of (3.136) is negative. Overall, we obtain an upper bound for (3.134) given by

$$2\Delta t \left(\frac{\delta'}{2\Delta t}\right)^{1-\alpha} + \frac{\delta'}{2}. \tag{3.137}$$

In addition, we have that $|R^\delta_{i,j}(a)| < |R^\delta_{i,j}(\tilde a)|$ since $m^\delta(\tilde a) = m^\delta(a) = 0$ and $M^\delta_{i,j}(a) = k^{i,j-1}_-(a) - K^{i,j} - \delta \le \tilde k^{i,j-1}_-(\tilde a) - \tilde K^{i,j} - \delta$. Hence, by collecting the above estimates and substituting to (3.133) we conclude that

$$\frac{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))}{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))} \le e^{\gamma^{-1}|I|\left(2\Delta t\left(\frac{\delta'}{2\Delta t}\right)^{1-\alpha}+\frac{\delta'}{2}\right)}.$$

In this case, $M(\gamma)$ is given by the exponent in the right hand side. As a general remark, we would like to stress that the above errors concern one space-time box, so the overall error should be multiplied by the total number of boxes. Furthermore, the changes in the given box influence all others as well and this has also to be taken into account, but the error is similar as the one computed here. So we do not detail it here.

<u>*Case 2:*</u> **The profile a exits the safety zone.** Similarly to *Case 1*, the new profile is $\tilde a_{i,j-1} := a_{i,j-1} - \delta'$ and $\tilde a_{i,j} := a_{i,j}$. We choose $\tilde x^+ = x^+ - \frac{\delta'}{4}$ and $\tilde k := k + \frac{\delta'}{2} > k$, i.e., we keep the same number of minus jumps and we decrease the number of plus jumps. Therefore, we have that

$$\tilde d = d + \frac{\delta'}{\Delta t} \quad \text{and} \quad |c_\pm(i,\tilde a) - c_\pm(i,a)| \le \beta\delta'|I|,$$

which implies that $|R^\delta_{i,j}(a)| \le |R^\delta_{i,j}(\tilde a)|$ since $m^\delta(a) \ge m^\delta(\tilde a)$ and $k^{i,j-1}_-(a) - K^{i,j}$ is smaller or equal than all $k^{i,j-1}_+(a)$, $k^{i,j-1}_+(\tilde a)$ and $k^{i,j-1}_-(\tilde a) - \tilde K^{i,j} - \delta$. Hence, using inequalities (3.135) and (3.136) as also the rates have been altered (in contrast to <u>Case 1</u>), we get the following upper bound for (3.134):

$$\Delta t\left(\frac{\delta'}{4\Delta t}\right)^{1-a} + 2\ln\left(1+\frac{\beta\delta'|I|}{c_m}\right) + 2\beta|I|\delta'\Delta t + \frac{\delta'}{2}.$$

Then, overall we have that

$$\frac{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))}{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))} \quad\le\quad e^{\gamma^{-1}|I|\left(\Delta t\left(\frac{\delta'}{4\Delta t}\right)^{1-a}+2\ln\left(1+\frac{\beta\delta'|I|}{c_m}\right)+2\beta|I|\delta'\Delta t+\frac{\delta'}{2}\right)}.$$

<u>*Case 3:*</u> **Both $a_{i,j-1}$ and $a_{i,j}$ are in the safety zone.** We subtract δ' from both $a_{i,j-1}$ and $a_{i,j}$, which also implies that $\tilde d = d$. Hence, we choose

$$\tilde x = x, \quad \tilde k = k,$$

which further implies that $|R^\delta_{i,j}(a)| \le |R^\delta_{i,j}(\tilde a|)$ and $|c_\pm(i,\tilde a) - c_\pm(i,a)| \le \beta\delta'|I|$. So the only terms in (3.134), (3.135) and (3.136) that contribute in the estimate are the terms which include the ratio and the difference of the rates. Thus, in this case, we obtain that:

$$\frac{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))}{\nu^i_{m_i((j-1)\Delta t)}(B^\delta_{i,j-1}(a))} \quad \leq \quad e^{\gamma^{-1}|I|\left(2\ln\left(1+\frac{\beta\delta'|I|}{c_m}\right)+2\beta|I|\delta'\Delta t\right)}.$$

With this we conclude the proof of Lemma 3.16 as $\gamma\frac{\epsilon^{-3}}{|I|\Delta t}M(\gamma) \lesssim \epsilon^{-3}(\eta_3^{1-\alpha}+\eta_3|I|) \to 0$ as $\gamma \to 0$. $\square$

Remark 3.20. In some realizations and some boxes, it may also happen that the number of plus or minus jumps is finite. We show that in such a case we can still work with profiles away from ± 1. Consider Case 1 with finite plus jumps when a is close to $+1$. The other cases can be done similarly. Then, in (3.127) for the probability of plus jumps $P_{\gamma^{-1}|I|c_+(i,a)}(N^-_{i,j-1} = n^{i,j-1}_-)$, as given in (3.76), we use the injective map ι as in Case 1 and obtain

$$\frac{e^{-\gamma^{-1}|I|\bar{c}_-(i,a)\Delta t}}{e^{-\gamma^{-1}|I|\bar{c}_-(i,\tilde{a})\Delta t}} \times \frac{(\gamma^{-1}|I|\bar{c}_-(i,a)\Delta t)^{(n^+_{i,j-1}+\frac{\gamma^{-1}|I|k^{i,j}}{2})}}{(\gamma^{-1}|I|\bar{c}_-(i,\tilde{a})\Delta t)^{(\tilde{n}^+_{i,j-1}+\frac{\gamma^{-1}|I|\tilde{k}^{i,j}}{2})}} \times \frac{(\tilde{n}^+_{i,j-1}+\frac{\gamma^{-1}|I|\tilde{k}^{i,j}}{2})!}{(n^+_{i,j-1}+\frac{\gamma^{-1}|I|k^{i,j}}{2})!} \leq$$

$$\leq \left(\gamma^{-1}|I|\Delta t \times \gamma^{-1}|I|\right)^{\gamma^{-1}|I|\frac{\delta'}{4}},$$

because the rates for a and $\tilde{a}$ are equal for the Case 1. Taking the logarithm of this error multiplied by the number of coarse-grained boxes, $\epsilon^{-3}/|I|\Delta t$, and multiplying by γ we get a vanishing number as $\gamma \to 0$:

$$\gamma\frac{\epsilon^{-3}}{|I|\Delta t}\gamma^{-1}|I|\frac{\delta'}{4}\ln\left(\gamma^{-1}|I|\Delta t \times \gamma^{-1}|I|\right),$$

since $\delta' = \Delta t \cdot \eta_3$ and $\eta_3 \cdot \epsilon^{-3} \to 0$.

Chapter 4

Action Minimisation and Macroscopic Interface Motion under Forced Displacement

4.1 The main result

Following [20], the cost defined in (3.36):

$$w_n(R,T) := n2\mathcal{F}(\bar{m}) + (2n+1)\left\{\frac{1}{\mu}\left(\frac{V}{2n+1}\right)^2 T\right\},$$

where $\mu =: 4\|\bar{m}'\|_{L^2(d\nu)}$ is the mobility coefficient. The first term is the cost of n nucleations while the second is the cost of displacement of $2n+1$ fronts (with the smaller velocity $V/(2n+1)$). Our main result is to prove Theorem 3.6, that is

Theorem. *Let $P > \inf_{n\geq 0} w_n(R,T)$.*

(i) Then $\forall\gamma > 0$ and for all sequences $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ with

$$I_{\Lambda_\epsilon\times\mathcal{T}_\epsilon}(\phi_\epsilon) \leq P,$$

we have:

$$\liminf_{\epsilon\to 0} I_{\Lambda_\epsilon\times\mathcal{T}_\epsilon}(\phi_\epsilon) \geq \inf_{n\geq 0} w_n(R,T) - \gamma,$$

where $w_n(R,T)$ is given in (3.36).

(ii) There exists a sequence $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ such that

$$\limsup_{\epsilon\to 0} I_{\Lambda_\epsilon\times\mathcal{T}_\epsilon}(\phi_\epsilon) \leq \inf_{n\geq 0} w_n(R,T).$$

We split the proof in the following sections: in Section 4.2 we first recall the notions of contours that allow us to separate the phases. Then we present the multi-instanton manifold and its properties. This is a repetition of [20] and the reader familiar with it could skip it. However, for completeness of the presentation we also include it here as we will need several of these concepts in the next sections. One of the key estimates in the proof is the fact that, because of the finite cost, the profiles can not be away from local equilibrium (instanton manifold) for too long as there is a driving gradient force pushing them back. The main ingredients for this are given in Section 4.2.3 and the key Proposition 4.8 is a bit different than [20], so its proof is adjusted to the new context. In Section 4.3.1 we outline the proof which consists in splitting the time into good/bad time intervals during which the cost is small/large, respectively. Moreover, we establish the fact that we cannot stay away from the instanton manifold for too long as the gradient dynamics drive us back. Hence, in good time intervals we will eventually find ourselves close to the instanton manifold and, once this happens, we stay there for the whole interval. Then, we can linearize around some instanton and attribute some velocity to each interface. This is presented in Section 4.3.5. Furthermore, we still need to "connect" the good time intervals between them and this will be explained in Section 4.3.6. On the other hand, during bad time intervals which are treated in Section 4.3.7, more interesting things can happen, namely creation of new fronts (nucleations). But due to the fact that the overall cost is finite, they cannot be too many and the overall displacement during the bad time intervals is negligible. Concluding, having split the cost into smooth displacement (with some velocity) and nucleations, we introduce a simplified, closer to macroscopic, model for the motion of the "centers" of the instantons. We call it "particle model" and analyze it in Section 4.4 concluding the proof of Theorem 3.6.

4.2 Preliminaries

In this section we recall some facts that we will use in the sequel. For a more complete exposition we refer the reader to the original paper [20] and to the monograph [60]. We start with the definition of contours and the Peierls estimates which are bounds on the spatial location of deviations from the equilibrium in terms of the energy $\mathcal{F}$.

4.2.1 Contours

Given $\ell > 0$, we denote by $\mathcal{D}^{(\ell)}$ the partition of $\mathbb{R}$ into the intervals $[n\ell, (n+1)\ell)$, $n \in \mathbb{Z}$, and by $Q_x^{(\ell)},\ x \in \mathbb{R}$ the interval containing x (note that x need not be the center of $Q_x^{(\ell)}$). We say that $Q_x^{(\ell)},\ Q_{x'}^{(\ell)}$ are connected, if the closures have nonempty intersection, i.e. $\overline{Q_x^{(\ell)}} \cap \overline{Q_{x'}^{(\ell)}} \neq \emptyset$. Now we define

$$m^{(\ell)}(x) := \frac{1}{|Q_x^{(\ell)}|} \int_{Q_x^{(\ell)}} m(y)\, dy. \tag{4.1}$$

Given an "accuracy parameter" $\zeta > 0$, we introduce

$$\eta^{(\zeta,\ell)}(m;x) = \begin{cases} \pm 1 & \text{if } |m^{(\ell)}(x) \mp m_\beta| \leq \zeta, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

where $\mp m_\beta$ are the solutions of the mean field equation (2.27). For any $\Lambda \subseteq \mathbb{R}$ which is $D^{(\ell)}$-measurable we call

$$\begin{aligned} \mathcal{B}_0^{(\zeta,\ell,\Lambda)}(m) &:= \left\{ x \in \Lambda :\ \eta^{(\zeta,\ell)}(m;x) = 0 \right\} \\ \mathcal{B}_\pm^{(\zeta,\ell,\Lambda)}(m) &:= \Big\{ x \in \Lambda :\ \eta^{(\zeta,\ell)}(m;x) = \pm 1,\ \text{there exists } x' \in \Lambda :\ \overline{Q_x^{(\ell)}} \cap \overline{Q_{x'}^{(\ell)}} \neq \emptyset \\ &\qquad \eta^{(\zeta,\ell)}(m;x') = -\eta^{(\zeta,\ell)}(m;x) \Big\}, \\ \mathcal{B}^{(\zeta,\ell,\Lambda)}(m) &:= \mathcal{B}_+^{(\zeta,\ell,\Lambda)}(m) \cup \mathcal{B}_-^{(\zeta,\ell,\Lambda)}(m) \cup \mathcal{B}_0^{(\zeta,\ell,\Lambda)}(m). \end{aligned}$$

Calling ℓ_- and ℓ_+ two values of the parameter ℓ, with ℓ_+ an integer multiple of ℓ_-, we define a "phase indicator"

$$\vartheta^{(\zeta,\ell_-,\ell_+)}(m;x) = \begin{cases} \pm 1 & \text{if } \eta^{(\zeta,\ell_-)}(m;\cdot) = \pm 1 \text{ in } \left(Q_{x-\ell_+}^{(\ell_+)} \cup Q_x^{(\ell_+)} \cup Q_{x+\ell_+}^{(\ell_+)} \right), \\ 0 & \text{otherwise,} \end{cases}$$

and call contours of m the connected components of the set $\{x :\ \vartheta^{(\zeta,\ell_-,\ell_+)}(m;x) = 0\}$. The interval $\Gamma = [x_-, x_+)$ is a plus contour if $\eta^{(\zeta,\ell_-)}(m;x_\pm) = 1$, a minus contour if $\eta^{(\zeta,\ell_-)}(m;x_\pm) = -1$, otherwise it is called mixed.

Moreover, for any measurable $\Lambda \subseteq \mathbb{R}$ and $m \in L^\infty(\mathbb{R} \to [-1,1])$, we define a local notion of energy by

$$\begin{aligned} \mathcal{F}(m_\Lambda | m_{\Lambda^c}) &:= \int_\Lambda \phi_\beta(x) dx + \frac{1}{4} \int_{\Lambda \times \Lambda} J(x,y)(m(x) - m(y))^2 dy\, dx \\ &\quad + \frac{1}{2} \int_{\Lambda \times \Lambda^c} J(x,y)(m(x) - m(y))^2 dy\, dx. \end{aligned}$$

The parameters (ζ, ℓ_-, ℓ_+) are called *compatible* with $(\zeta_0, c_1, \kappa) \in \mathbb{R}^3_+$ if $\zeta \in (0, \zeta_0)$, $\ell_- \le \kappa\zeta$, $\ell_+ \ge 1/\ell_-$, and if for any $D^{(\ell_-))}$-measurable set Λ and any $m \in L^\infty(\mathbb{R} \to [-1,1])$

$$\mathcal{F}(m_\Lambda | m_{\Lambda^c}) \ge c_1 \zeta^2 |\mathcal{B}^{(\zeta, \ell_-, \Lambda)}(m)|.$$

With the above definitions we have:

Theorem 4.1 ([6]). *There are positive constants ζ_0, c_1, κ, c_2, so that if (ζ, ℓ_-, ℓ_+) is compatible with (ζ_0, c_1, κ), then for all $m \in L^\infty([-L, L]; [-1,1])$,*

$$\mathcal{F}(m) \ge \sum_{\Gamma \text{ contour of } m} w_{\zeta, \ell_-, \ell_+}(\Gamma), \tag{4.3}$$

where

$$w_{\zeta,\ell_-,\ell_+}(\Gamma) = c_1 \zeta^2 \frac{\ell_-}{\ell_+} |\Gamma|, \text{ if } \Gamma \text{ is a plus or a minus contour;}$$

$$w_{\zeta,\ell_-,\ell_+}(\Gamma) = \max \left\{ c_1 \zeta^2 \tfrac{\ell_-}{\ell_+} |\Gamma| \ ; \ \mathcal{F}(\bar{m}) - c_2 e^{-\alpha \ell_+} \right\}, \text{ if } \Gamma \text{ is a mixed contour}$$

and α is given in (3.15).

From [18] we have that:

$$I_{[t_0,t_1]}(\phi) \ge \frac{\beta}{2} (\mathcal{F}(\phi(\cdot, t_1)) - \mathcal{F}(\phi(\cdot, t_0))) + \int_{t_0}^{t_1} \| 1 \wedge |f(\phi)| \|_2^2 \, dt. \tag{4.4}$$

Formulas (4.4) and (3.37) yield

$$\sup_{t \le \epsilon^{-2} T} (\mathcal{F}(\phi_\epsilon(\cdot, t)) - \mathcal{F}(\phi_\epsilon(\cdot, 0))) \le P, \tag{4.5}$$

for every ϕ_ϵ in $\mathcal{U}[\epsilon^{-1} R, \epsilon^{-2} T]$. Then, by Theorem 4.1, for ζ small enough,

$$\sum_{\Gamma_i \text{ contours of } u(\cdot, t)} |\Gamma_i| \ \le \ \frac{\ell_+}{c_1 \ell_-} \zeta^{-2} (P + F(\bar{m})) \tag{4.6}$$

$$\text{number of contours of } u(\cdot, t) \le \frac{1}{c_1 \ell_-} \zeta^{-2} (P + F(\bar{m})) =: N_{\max} \tag{4.7}$$

$$\text{number of mixed contours of } u(\cdot, t) \le \frac{P + F(\bar{m})}{\mathcal{F}(\bar{m}) - c_2 e^{-\alpha \ell_+}} =: N_{\max}^{\text{mix}} \tag{4.8}$$

4.2.2 Multi-instanton manifold

The instanton manifold is the set $\mathcal{M}^{(1)} = \{\bar m_\xi, \xi \in \mathbb{R}\}$. We extend the notion to the case of several coexisting instantons by defining the multi-instanton manifold $\mathcal{M}^{(k)}$, $k > 1$, as the set of all $\bar m_{\bar\xi}$, $\bar\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$, $\xi_1 < \dots < \xi_k$, sufficiently apart from each other such that, setting $\xi_0 := -\infty,\ \xi_{k+1} := \infty$, the function

$$\bar m_{\bar\xi}(x) := \begin{cases} \bar m(x - \xi_j) & \text{if } x \in \left[\frac{\xi_{j-1}+\xi_j}{2}, \frac{\xi_{j+1}+\xi_j}{2}\right] \text{ and } j \text{ odd,} \\ \bar m(\xi_j - x) & \text{if } x \in \left[\frac{\xi_{j-1}+\xi_j}{2}, \frac{\xi_{j+1}+\xi_j}{2}\right] \text{ and } j \text{ even,} \end{cases}$$

has exactly k mixed contours. We denote

$$\mathcal{M} = \bigsqcup_{k \ge 1} \mathcal{M}^{(k)}. \tag{4.9}$$

To study "neighborhoods" of $\mathcal{M}$ we introduce the notion of "center of m" that we use here in a slightly different sense than usual:

Definition 4.2. Recalling $L^2(d\nu_\xi)$, the point $\xi \in \mathbb{R}$ is a center of m if $\xi \in \Gamma$, Γ a mixed contour of m, and if

$$\big(m - \bar m_\xi, \bar m'_\xi\big)_{L^2(d\nu_\xi)} = 0, \quad \text{or, equivalently,} \quad \big(m, \bar m'_\xi\big)_{L^2(d\nu_\xi)} = 0. \tag{4.10}$$

where $\bar m'_\xi$ is the derivative with respect to space of $\bar m_\xi$. ξ is an odd, even, center if Γ is a $(-,+)$, respectively $(+,-)$ mixed contour.

The following theorem holds, see [28],

Theorem 4.3. *If ζ (in the definition of contours) is small enough the following holds.*

- *Each mixed contour Γ of m contains a center of m.*
- *There is $\delta > 0$ so that if for some ξ in a $(-,+)$ mixed contour Γ of m (analogous statement holding in the $(+,-)$ case), $\|\mathbf{1}_\Gamma(m - \bar m_\xi)\|_{L^2(d\nu_\xi)} \le \delta$, then there is a unique center ξ_m in Γ and*

$$\int_{\mathbb{R}} \Big(\{m - \bar m_{\xi'}\}^2 - \{m - \bar m_{\xi_m}\}^2 \Big) > 0, \quad \textit{for all } \xi' \in \Gamma,\ \xi' \neq \xi_m \tag{4.11}$$

and calling $v = m - \bar m_\xi$, $N_{v,\xi} = \dfrac{(v, \bar m'_\xi)}{(\bar m', \bar m')}$,

$$\big|\xi_m - (\xi - N_{v,\xi})\big| \le c\|v\|^2_{L^2(d\nu_\xi)}, \qquad |N_{v,\xi}| \le c\|v\|_{L^2(d\nu_\xi)}. \tag{4.12}$$

- *If also $\inf_{\xi'} \|\mathbf{1}_\Gamma(n - \bar m_{\xi'})\|_{L^2(d\nu_{\xi'})} \le \delta$ and $\|m - n\|_{L^2(d\nu_\xi)}$ is small, then*

$$|\xi_m - \xi_n| \le c\|m - n\|_{L^2(d\nu_\xi)}. \tag{4.13}$$

In Sect. 4.6 we will prove the third statement for both the L^1 and the L^2 norm. By the first statement in Theorem 4.3 a function m with k mixed contours $\Gamma_1, .., \Gamma_k$ has (at least) one center in each one of the mixed contours; we denote by Ξ the collection of all $\bar{\xi} = (\xi_1, .., \xi_k)$, $\xi_i < \xi_{i+1}$, ξ_i a center of m in Γ_i and define

$$d_{\mathcal{M}}(m) = \inf_{\bar{\xi} \in \Xi} \ \|m - \bar{m}_{\bar{\xi}}\|_{L^2(d\nu_{\bar{\xi}})}. \tag{4.14}$$

If m is close enough to $\mathcal{M}^{(k)}$, then the choice of $\bar{\xi}$ is unique. Note that this definition differs slightly from the usual definition of a distance of a point from a manifold, but the following lemma bounds this difference by replacing the $\inf$ over centers in (4.14), by the $\inf$ over any generic variable $\bar{\xi} \in \Gamma_1 \times .. \times \Gamma_k$, with $\bar{\xi} = (\xi_1, \ldots, \xi_k)$:

Lemma 4.4. *For all $k \in \mathbb{N}$ there are $\delta > 0$ and $c > 0$ so that if m has k mixed contours $\Gamma_1, .., \Gamma_k$ and $d_{\mathcal{M}}(m) \leq \delta$, then*

$$d^2_{\mathcal{M}}(m) \ \geq \inf_{\bar{\xi} \in \Gamma_1 \times .. \times \Gamma_k} \ \|m - \bar{m}_{\bar{\xi}}\|^2_{L^2(d\nu_{\xi})} \ \geq \ d^2_{\mathcal{M}}(m) - c \sum_{i=1}^{k-1} e^{-\alpha \, \mathrm{dist}(\Gamma_{i+1}, \Gamma_i)/2}, \tag{4.15}$$

where $\alpha > 0$ is defined in (3.15).

For the proof we refer to [20].

4.2.3 Permanence away from equilibrium

In this section we get bounds on the time interval when a profile is away from the multi-istanton manifold. This is done by obtaining a lower bound on the energy gradient in terms of the distance from the manifold and we will use it in Theorem 4.13 in order to get a bound on the number of time intervals where the given profile is away from local equilibrium. The main theorem is:

Theorem 4.5. *For any $\vartheta > 0$ there is $\rho > 0$ such that the following holds. Let $m \in L^\infty(\mathbb{R}; (-1, 1))$ have an odd number p of mixed contours, let $\mathcal{F}(m) \leq P$ (P as in Theorem 3.6) and let $d_{\mathcal{M}}(m)^2 \geq \vartheta$. Then*

$$\int_{\mathbb{R}} (1 \wedge |f(m)|)^2 \geq \rho, \tag{4.16}$$

where f is defined in (3.12).

The proof is essentially contained in [20]. Here we only present the necessary modifications needed for the new functional. This theorem implies a penalization of the time away local from equilibrium which is stated in the following corollary:

Corollary 4.6. *Let ϕ satisfy* (3.37)*, then for any $\vartheta > 0$ there is $c_{4.6} > 0$ and $\rho > 0$ so that, if $d_{\mathcal{M}}(\phi(\cdot, t)) \geq \vartheta$ when $t \in [t_0, t_1]$, $0 \leq t_0 < t_1 \leq \epsilon^{-2}T$, then necessarily $t_1 - t_0 \leq \frac{3P}{c_{4.6}\rho}$.*

Proof. By recalling (4.5) and from Theorem 4.5 we obtain that for some $c_{4.6} > 0$

$$3P \geq c_{4.6} \int_{t_0}^{t_1} \|1 \wedge |f(\phi)|\|_2^2 \, dt \geq c_{4.6}\rho \, (t_1 - t_0),$$

which concludes the proof. □

Now we argue as in [20]. We start with the analysis of the condition $d_{\mathcal{M}}(m)^2 \geq \vartheta$ when the deviation of m from $\bar{m}_{\xi}$ is localized in a neighborhoud of the contours. We first give the necessary notation. Let Q, Q_j and $B^{\pm}_{k,j}$ be intervals of the form $Q = [a, b)$, $Q_j = [a - j, b + j)$, $B^{-}_{k,j} = [a - j - k, a - j)$, $B^{+}_{k,j} = [b + j, b + j + k)$ with a, b, j, k all in $\ell_+\mathbb{N}$. Then, given $\vartheta > 0$, we set

$$U_{Q,j,\vartheta} = \Big\{ m \in L^{\infty}(\mathbb{R}, (-1, 1)) : \ Q \text{ is a mixed} \pm \text{contour for } m \\ \text{and} \inf_{\xi \in Q} \int_{Q_j} |m - \bar{m}_{\xi}|^2 \geq \vartheta \Big\} \tag{4.17}$$

and

$$V_{k,j} = \Big\{ m \in L^{\infty}(\mathbb{R}, (-1, 1)) : \ \eta^{(\zeta, \ell_-)}(m; x) = \pm 1 \text{ for all } x \in B^{\pm}_{k,j} \Big\}. \tag{4.18}$$

Lemma 4.7. *For any $\vartheta > 0$, Q and Q_j as above, there is k so that*

$$\int_{Q_{k+j}} |f(m)| > 0 \qquad \textit{for any } m \in U_{Q,j,\vartheta} \cap V_{k,j}. \tag{4.19}$$

The proof is given in [20]. With this lemma we can prove the following:

Proposition 4.8. *For any $\vartheta > 0$, Q and Q_j, let k be as in Lemma 4.7. Then there is $\rho > 0$ so that*

$$\inf_{m \in U_{Q,j,\vartheta} \cap V_{k,j}} \int_{Q_{k+j}} |1 \wedge |f(m)||^2 \geq \rho. \tag{4.20}$$

Proof. Suppose that the opposite is true. Then there exists a sequence $m_n \in U_{Q,j,\vartheta} \cap V_{k,j}$ such that

$$\lim_{n \to \infty} \int_{Q_{k+j}} |1 \wedge |f(m_n)||^2 = 0,$$

which implies that $|A_n^c| \to 0$ and $\int_{Q_{k+j}\cap A_n} |f(m_n)|^2 \to 0$ where $A_n := \{x : \ |f(m_n(x))| < 1\}$. We also have that $m_n \rightharpoonup \hat{m}$ in $L^2_{\rm loc}$ and hence $J * m_n \to J * \hat{m}$ in $L^2_{\rm loc}$. We write (recall that $f(m) = J * m - \operatorname{arctanh} m$):

$$
\begin{aligned}
m_n &= m_n \mathbf{1}_{A_n} + m_n \mathbf{1}_{A_n^c} = \tanh(J * (m_n \mathbf{1}_{A_n}) - f(m_n \mathbf{1}_{A_n}))\mathbf{1}_{A_n} + m_n \mathbf{1}_{A_n^c} \\
&= \tanh(\beta J * m_n - f(m_n)\mathbf{1}_{A_n})\mathbf{1}_{A_n} + m_n \mathbf{1}_{A_n^c}.
\end{aligned} \tag{4.21}
$$

Then, $\|m_n\|_\infty \le 1$ implies that $m_n \mathbf{1}_{A_n^c} \to 0$ in L^2. For the first term of m_n in (4.21) we have:

$$
\begin{aligned}
\int_{Q_{k+j}} |m_n \mathbf{1}_{A_n} - \tanh(\beta J * \hat{m})|^2 &\le \int_{Q_{k+j}\cap A_n} |\tanh(\beta J * m_n - f(m_n)) - \tanh(\beta J * \hat{m})|^2 \\
&\le c \int_{Q_{k+j}\cap A_n} |f(m_n)|^2 \to 0,
\end{aligned}
$$

since $\tanh$ is uniformly Lipschitz continuous. Thus, $\lim_{n\to\infty} m_n = \tanh(\beta J * \hat{m})$ in $L^2(Q_{k+j})$. Therefore, since both $m_n \rightharpoonup \hat{m}$ in $L^2_{\rm loc}$ and $m_n \to \tanh(\beta J * \hat{m})$ in Q_{k+j} we obtain that

$$
\hat{m} = \tanh(\beta J * \hat{m}) \text{ in } Q_{k+j} \ \text{ and } \ f(\hat{m})(x) = 0 \ \forall x \in Q_{k+j}.
$$

Now we obtain the contradiction. We have that

$$
\inf_{\xi \in Q} \int_{Q_j} |m_n - \bar{m}_\xi|^2 \ge \vartheta, \ \ \forall n,
$$

which implies (since $\lim_{n\to\infty} m_n = \tanh(\beta J * \hat{m})$ in $L^2(Q_{k+j})$) that

$$
\inf_{\xi \in Q} \int_{Q_j} |\tanh(\beta J * \hat{m}) - \bar{m}_\xi|^2 \ge \vartheta,
$$

which (since $\hat{m} = \tanh(\beta J * \hat{m})$ in Q_{k+j}) in turn implies that $\hat{m} \in U_{Q,j,\vartheta}$. Furthermore, $\hat{m} \in V_{k,j}$ (closed in weak L^2). Thus, by lemma 4.7 there exists k^* such that $\int_{Q_{k+j}} |f(m)| > 0$ for all $m \in U_{Q,j,\vartheta}$. Contradiction, since this is not true for $\hat{m}$. $\square$

A similar result is true when the external conditions are in the plus or minus phase. Let

$$
U^\pm_{Q,j,\vartheta} = \Big\{ m \in L^\infty(\mathbb{R}, (-1,1)) : Q \text{ is a } \pm \text{ contour for } m \\
\text{and } \int_{Q_j} |m \mp m_\beta|^2 \ge \vartheta \Big\} \tag{4.22}
$$

$$
V^\pm_{k,j} = \Big\{ m \in L^\infty(\mathbb{R}, (-1,1)) : \ \eta^{(\zeta,\ell_-)}(m;x) = \pm 1 \text{ for all } x \in B^-_{k,j} \cup B^+_{k,j} \Big\}. \tag{4.23}
$$

Then we also have the following:

Proposition 4.9. *For any $\vartheta > 0$, Q and Q_j there are k and $\rho > 0$ so that*

$$\inf_{m \in U^{\pm}_{Q,j,\vartheta} \cap V^{\pm}_{k,j}} \int_{Q_{k+j}} (1 \wedge |f(m)|)^2 \geq \rho. \tag{4.24}$$

With these ingredients we can conclude the proof of Theorem 4.5 following [20].

4.3 Proof of Theorem 3.6

4.3.1 Good and bad time intervals

Given $\epsilon > 0$, we fix an orbit $\phi \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ as in Theorem 3.6 (neglecting from the notation the dependence on ϵ) and let $b(\phi)$ in (3.18) be the external force to which it corresponds. We decompose the time interval $[0,\, \epsilon^{-2}T]$ into subintervals $\{S[j, j+1),\ j \in \mathbb{N}\}$ of length $S > 0$. For $\kappa > 0$ we choose a parameter

$$\delta \equiv \delta(\epsilon) := |\log \epsilon|^{-\kappa} \tag{4.25}$$

and define

$$\phi^{(\delta,S)}(\phi; t) = \begin{cases} 1, & \text{if } \displaystyle\int_{jS}^{(j+1)S} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x,t) dx\, dt < \delta \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } t \in S[j, j+1). \tag{4.26}$$

To construct "time contours" we also define $\Phi^{(\delta,S)}(\phi; t)$ equal to 1 if $\phi^{(\delta,S)}(\phi; s) = 1$ for all $s \in S[j-1, j+1)$ and $= 0$ otherwise. We define $G_{\text{tot}} = \{t \leq \epsilon^{-2}T : \Phi^{(\delta,S)}(\phi; t) = 1\}$ and call t a "good time" and $S[j, j+1)$ a "good time interval" if they are contained in G_{tot}. Bad times and bad intervals are defined complementary.

Given the fact that it is too expensive to be away from the instanton manifold (Corollary 4.6), the strategy now is to relate the cost functional to the cost of two mechanisms: translation of the interfaces and nucleation of new ones. The first can be achieved by relating the cost to the driving force of the motion of the interface and subsequently to its velocity. This is a valid approximation during the "good" time intervals. On the other hand, nucleations can only happen in the "bad" ones during which, the already existing interfaces cannot move too much because the overall cost is finite. We quantify all this in the next sections. We introduce the velocity of the formed interfaces and relate it to the cost. Contrary to [20], for the case of the cost derived via the large deviations this is not straightforward and new auxiliary profiles have to be introduced.

4.3.2 Parameters of the proof.

We start by choosing some crucial parameters in the estimates. In Theorem 3.1 we saw that the cost of a nucleation (producing two fronts) is close to the cost of creating two interfaces, i.e., close to $2\mathcal{F}(\bar{m})$. Since the total cost is bounded by P, we obtain an upper bound (n^*) on the total number of fronts:

$$n^* = 1 + \frac{2P}{\mathcal{F}(\bar{m})}. \tag{4.27}$$

Moreover, following [20], for given $\gamma > 0$ we choose a critical value ℓ^* for the displacement of the fronts, after which we consider that a nucleation has occurred. This is determined to be such that the following holds:

$$\left|\mathcal{F}\big(\bar{m}_{(-\ell^*,\ell^*)}\big) - 2\mathcal{F}(\bar{m})\right| \le \gamma, \qquad \text{where} \quad \bar{m}_{(-\ell^*,\ell^*)} = \mathbf{1}_{x\ge 0}\bar{m}_{\ell^*} - \mathbf{1}_{x<0}\bar{m}_{-\ell^*}. \tag{4.28}$$

This means that if the profile is made out of a combination of instantons whose centers are far enough (more than $2\ell^*$) then its free energy is well approximated by the number of such instantons times the cost of each one of them. Indeed, by the L^2-continuity of $\mathcal{F}(\cdot)$, there is $\vartheta > 0$ so that for all m such that $d_{\mathcal{M}}(m) \le \vartheta$ and with centers $(\xi_1, .., \xi_n)$, $n \le n^*$, $\xi_{i+1} - \xi_i \ge 2\ell^*$, $\forall i$, we have that:

$$\left|\mathcal{F}(m) - n\mathcal{F}(\bar{m})\right| \le n^*\gamma. \tag{4.29}$$

However, it may happen that in a newly created nucleation the centers do not exceed the distance $2\ell^*$. These are called "incomplete nucleations" and we can neglect them arguing as in [20], [6] and [7] using the propositions below.

We first note that starting with such a profile, the free dynamics make it disappear within a finite time, depending on the distance ℓ (see [6], Proposition 7.1):

Proposition 4.10. *There is $\tau > 0$ so that for any positive $\ell \le \ell^*$, the solution $v(x,s)$ of* (3.10) *starting from $\bar{m}_{(-\ell,\ell)}$ (as defined in* (4.28)*) verifies*

$$\sup_{x\in\mathbb{R}} |v(x,\tau) - m_\beta| \le \vartheta.$$

This can be also used in a multi-instanton setting:

Proposition 4.11. *There is $L > 0$ for which the following holds. Let ℓ and τ be as in Proposition 4.10 and $\bar{\xi} = (\xi_1, ..., \xi_n)$, $n \le n^*$. Call $\mathcal{I}$ the set of all even i such that*

$\xi_{i+1} - \xi_i \le \ell$. *Suppose* $\mathcal{I}$ *non void and that for* $j \notin \mathcal{I}$, $\xi_{j+1} - \xi_j \ge L$. *Then the solution* $w(x,t)$ *of* (3.10) *which starts from* $\bar{m}_{\bar{\xi}}$ *is such that*

$$\sup_{x\in\mathbb{R}} |w(x,\tau) - \bar{m}_{\bar{\xi}^*}(x)| \le \vartheta, \tag{4.30}$$

where $\bar{\xi}^*$ *is obtained from* $\bar{\xi}$ *by dropping all pairs* ξ_i, ξ_{i+1}, $i \in \mathcal{I}$.

Then, the same is true if we have an external force whose cost is controlled by a parameter $\alpha > 0$.

Proposition 4.12. *Let* ℓ, τ, L, $\bar{\xi}$ *and* $\bar{\xi}^*$ *as previously. Then there is* $\alpha > 0$ *such that if*

$$\|m - \bar{m}_{\bar{\xi}}\|_2 \le \vartheta, \qquad \int_0^\tau \int_{\mathbb{R}} |b(x,t)|^2 \, dx \, dt \le \alpha, \tag{4.31}$$

then the solution $w(x,t)$ *of* (3.19) *with force* b *and which starts from* m *is such that*

$$\|w(x,\tau) - \bar{m}_{\bar{\xi}^*}(x)\|_2 \le 4\vartheta. \tag{4.32}$$

From the previous propositions, we fix the parameters S and δ of our problem. Following the analysis in [20] we first choose the parameter S to be of order one such that:

$$S > 10^3 \max\big\{\tau, \frac{3P}{c_{4.6}\rho}, \frac{4}{\omega}\big\}, \tag{4.33}$$

where ω is the spectral gap parameter given in Section 4.3.5. On the other hand, for δ a safe choice would be

$$\delta = 10^{-3} \min\big\{\alpha, \frac{\vartheta}{c_{4.14}}\big\}, \quad \alpha \text{ and } c_{4.14} \text{ as in Proposition 4.12 and Proposition 4.14} \tag{4.34}$$

Hence, our choice in (4.25) satisfies the above criteria. With this choice of S and δ we have the following theorem:

Theorem 4.13. *Let* ϕ *satisfy* (3.37) *and let* δ *and* S *as above. Then:*

$$\textit{number of bad time intervals} \le \frac{2P}{\delta}. \tag{4.35}$$

If $S[j, j+1)$ *is a good time interval, then there is* $t_1 \in S[j - \frac{1}{2}, j - \frac{1}{4})$ *such that* $d_{\mathcal{M}}(\phi(\cdot, t_1)) \le \vartheta$.

Proof. Suppose that I is a bad interval and let I^- be its previous. Then inequality (4.26) cannot hold for both I and I^- since otherwise I would have been a good interval. Hence, the number of bad intervals is at most twice the number of intervals where (4.26) is not true. Thus,

$$P > \sum_{I:\,(4.26)\text{ is true}} + \sum_{I:\,(4.26)\text{ not true}} > \frac{1}{2}(\#\text{bad intervals})\delta$$

The second statement follows from Corollary 4.6. □

4.3.3 Construction of auxiliary profiles ϕ_1 and m.

Theorem 4.13 allows us to find times $t_j \in [j - \frac{1}{2}, j - \frac{1}{4}]S$, $j \in \mathcal{J} := \{1, 2, \ldots, \frac{\epsilon^{-2}T}{S}\}$ for every good time interval $S[j, j+1)$, such that $d_{\mathcal{M}}(\phi(\cdot, t_j)) \le \vartheta$. Then we define a new partition of $[0, \epsilon^{-2}T]$ as follows: if $S[j, j+1)$ is a good time interval in the original partition, we replace it by $[t_j,\, t_{j+1})$ and modify the neighbouring bad time intervals accordingly. For example, if the previous is bad, in the new partition it will be replaced by $[S(j-1),\, t_j)$. If $S[j+1,\, j+2)$ is a good time interval as well, then t_{j+1} are the ones given by Theorem 4.13, otherwise, $t_{j+1} := S(j+1)$. In this way, we obtain a new, slightly shifted, partition $\{[t_j, t_{j+1})\}_{j\in\mathcal{J}}$ of $[0, \epsilon^{-2}T]$. Note that in the new partition, the bad time intervals remain unchanged and this will be relevant in Section 4.3.7.

To prove Theorem 3.6, we want to derive lower bounds to the cost for a given profile given the condition on the total displacement. We estimate the cost of the given profile by assigning a notion of velocity to its fronts. The total displacement is then related to the motion of these fronts with the assigned velocity. We implement these during the good time intervals.

Suppose t_j is the left endpoint of a maximal connected component G of G_{tot}. By the definition of t_j we have that $d_{\mathcal{M}}(\phi(\cdot, t_j)) \le \vartheta$. For ϑ small enough, ϕ has only mixed contours which we denote by $\{\Gamma_i\}_{i=1}^k$, for some k odd. We call $\bar{\xi} = (\xi_1, \ldots, \xi_k)$ its centers, ordered increasingly. In the first good time interval $[t_j,\, t_{j+1})$ of the connected component G, we construct an approximate (to ϕ) profile ϕ_1 as well as another orbit m as follows: First we truncate the forcing term $b(\phi)$. For $\lambda > 0$ we choose a threshold

$$\Delta \equiv \Delta(\epsilon) := |\log \epsilon|^{-\lambda}, \qquad \lambda < \kappa, \tag{4.36}$$

for $\kappa > 0$ as in (4.25), and define a new external field

$$b_1(x,t) := b(\phi)(x,t)\mathbf{1}_{\{(x,t):\,|b(\phi)(x,t)|\le\Delta(\epsilon)\}}. \tag{4.37}$$

Then we define the auxiliary profiles ϕ_1 and m to be the solutions of the following system:

$$\frac{d}{dt}\phi_1 = -\phi_1 + \tanh(\beta J * \phi_1) + \alpha b_1, \qquad \phi_1(\cdot, t_{\text{in}}^+) = \phi(\cdot, t_{\text{in}}^+), \tag{4.38}$$

where

$$\alpha(x,t) := \left(\frac{1 - \bar{m}^2_{\tilde{\xi}(t)}}{8} \right)^{1/2}. \tag{4.39}$$

The approximate centers $\tilde{\xi}(t)$, defined in (4.44), are the centers of the profile m that satisfies the equation:

$$\frac{d}{dt}m = -m + \tanh(\beta J * m) + b(\phi_1), \qquad m(\cdot, t_{\text{in}}^+) = m^{\text{in}}(\cdot). \tag{4.40}$$

Recall the definition of function b given in (3.18). The time t_{in} and the initial condition $m^{\text{in}}(\cdot)$ are given below. For simplicity of the notation we drop in t_{in} the dependence on j. Note that for the coefficient $\alpha(x,t)$ defined in (4.39) there exists a large constant $c_* > 0$ such that

$$\frac{1}{c_*} \leq \alpha(x,t) \leq 1, \quad \forall x, t. \tag{4.41}$$

Existence and uniqueness of solutions of the system (4.38)-(4.40) is proved in Sect. 4.5. The idea for introducing the new force b_1 is that, following Sect. 4.7, for forces of order $\Delta(\epsilon)$, the density $\mathcal{H}$ of the cost is well approximated by b^2. Moreover, an extra factor $\alpha(x,t)$ is needed in order to reconcile the coefficient of the asymptotics of $\mathcal{H}$ (see (3.24)) with the space $L^2(\mathbb{R}, d\nu_{\bar{\xi}})$ in which we will be working later for the linearization around a moving instanton. Hence, the reason of introducing ϕ_1 is to have a profile whose centers are in a controlled distance from those of ϕ and additionally it has an external force which can be estimated by the cost. Then we use the idea in [20] of constructing sub-solutions (in our case of ϕ_1 rather than of ϕ) which start from an appropriately “regularized" initial profile and whose centers are ensured to move (being sub-solutions) at least as fast as the corresponding of ϕ. We denote this profile by m and note that, by a comparison theorem (see Theorem [27] or Theorem 2.29), it holds that $m(x,t) \leq \phi_1(x,t)$ for $x \in \mathbb{R}$ and $t \in [t_j,\, t_{j+1})$. Next we present the initial condition $m^{\text{in}}(\cdot)$ by following the initialization procedure described in [20], Section 10.

4.3.4 Initial condition

We work in the first good time interval $[t_j, t_{j+1})$. Given $m(\cdot, t_j)$ from equation (4.77), we construct $m^{\text{in}}(\cdot)$ as follows. Let $\bar{\xi}(m) = (\xi_1(m), \ldots, \xi_k(m))$ be the centers of m at time

t_j.

<u>Case 1</u>: When $\xi_{j+1}(m) - \xi_j(m) > 2|\log \epsilon|^2$ for all j. We let $t_{\text{in}} = t_j$ and $m(\cdot, t_{\text{in}}^+) = m(\cdot, t_{\text{in}}^-)$.

<u>Case 2</u>: When $\xi_{j+1}(m) - \xi_j(m) \le 2|\log \epsilon|^2$ for some j odd. We erase both centers for those j's and we call the new configuration $\bar{\xi}^{(1)}(m)$, for which it holds that $\bar{m}_{\bar{\xi}^{(1)}(m)} \le \bar{m}_{\bar{\xi}(m)}$. Then, we look at all even j in $\bar{\xi}^{(1)}(m)$ such that $2\ell^* \le \xi_{j+1}(m) - \xi_j(m) \le 2|\log \epsilon|^2$, ℓ^* as in Proposition 4.12 and we move each $\xi_j(m)$, $\xi_{j+1}(m)$ to $\xi'_j(m)$, $\xi'_{j+1}(m)$ so that

$$\xi_j(m) + \xi_{j+1}(m) = \xi'_j(m) + \xi'_{j+1}(m), \quad \xi'_{j+1}(m) - \xi'_j(m) = 2|\log \epsilon|^2.$$

We call $\bar{\xi}^{(2)}(m)$ the new configuration and $\bar{\xi}^{(3)}(m)$ the one obtained by $\bar{\xi}^{(2)}(m)$ following the same procedure as to obtain $\bar{\xi}^{(1)}(m)$. In $\bar{\xi}^{(3)}(m)$ the pairs $\xi_j(m)$, $\xi_{j+1}(m)$ with j even either satisfy $\xi_{j+1}(m) - \xi_j(m) \ge 2|\log \epsilon|^2$ or $\xi_{j+1}(m) - \xi_j(m) \le 2\ell^*$. <u>Case 2</u> is when $\xi_{j+1}(m) - \xi_j(m) \ge 2|\log \epsilon|^2$ for all j. Then, we define

$$\tilde{m}(x, t_j) = \min\{m(x, t_j), \bar{m}_{\bar{\xi}_3(m)}\},$$

$t_{\text{in}} = t_j$ and $m(\cdot, t_{\text{in}}) = \tilde{m}(\cdot, t_j)$.

<u>Case 3</u>: This case covers all remaining possibilities in the previous case when in $\bar{\xi}^{(3)}(m)$ there is at least a pair $\xi_j(m)$, $\xi_{j+1}(m)$ with j even satisfying $\xi_{j+1}(m) - \xi_j(m) \le 2\ell^*$. In that case, we let $t_{\text{in}} = t_j + \tau$, τ as in Proposition 4.12 and $m(\cdot, t_{\text{in}}^+)$ is the solution at time $t_j + \tau$ of (3.10) starting from $\tilde{m}(x, t_j)$. We finally define $m^{\text{in}}(\cdot) := m(\cdot, t_{\text{in}})$.

If $j = 0$ (and hence $t_j = 0$), $m(\cdot, 0)$ is the instanton $\bar{m}(\cdot)$, and initialization is not needed.

As a result of this initialization procedure, we have that for all $\epsilon > 0$ small enough, the centers of $m(\cdot, t_j)$ have mutual distance $\ge |\log \epsilon|^2$ and $d_{\mathcal{M}}(m(\cdot, t_{\text{in}}^+)) \le 6\vartheta$. To prove this, we use Proposition 4.12 with external force $b := b(\phi_1) = \alpha b_1$. In such a case, we have that $\int b^2$ is related to the cost since we apply it within a good time interval; hence the requirement (4.31) is satisfied. In the next section we show that in the good time interval $[t_j, t_{j+1})$ the solution $m(t, \cdot)$ of (4.40) follows closely a moving instanton $\bar{m}_{\bar{\xi}(t)}$, where $\bar{\xi}(t)$ are the centers of $m(t, \cdot)$.

4.3.5 Linearization around a moving instanton

By the constuction in the previous section, we have that in the good time interval $[t_j,\, t_{j+1})$ the profile m solves the equation

$$\frac{d}{dt}m = -m + \tanh(\beta J * m) + b(\phi_1), \qquad m(\cdot, t_j) = m^{\text{in}}(\cdot), \tag{4.42}$$

where the initial condition $m^{\text{in}}(\cdot)$ is given by the same initialization as in [20], i.e., it has an odd number k of mixed contours at mutual distance $\geq\ |\log \epsilon|^2$; moreover $d_{\mathcal{M}}(m^{\text{in}}(\cdot)) \leq 6\vartheta$.

Choice of parameters. From [28] we recall that there exists $\omega > 0$ such that

$$(v, Lv)_{L^2(d\nu)} \leq -\omega \|v\|_{L^2(d\nu)}, \tag{4.43}$$

for every $v \in L^2(d\nu)$ with $(v, \bar{m}')_{L^2(d\nu)} = 0$, where L is the linearized operator of the evolution (3.10). This is called "spectral gap parameter" . Moreover, let c be given in (4.52) and $\epsilon_1 < \frac{\omega}{8c}$. Calling $\bar{\xi}(t) = (\xi_1(t), .., \xi_k(t))$ the centers of $m(\cdot, t)$, $t \geq t_j$, we define the approximate centers $\tilde{\xi}(t) = (\tilde{\xi}_1(t), .., \tilde{\xi}_k(t))$ and the deviation $u(\cdot, t)$ as follows:

$$\big(\mathbf{1}_{A_{\alpha^*}} \bar{m}'_{\tilde{\xi}_i(t)}, [m(\cdot, t) - \sigma_i \bar{m}_{\tilde{\xi}_i(t)}]\big)_{L^2(d\nu)} = 0, \qquad u(\cdot, t) = m(\cdot, t) - \bar{m}_{\tilde{\xi}(t)}, \tag{4.44}$$

where

$$A_{\alpha^*} := \left\{ x \in \mathbb{R} : \int_{t_{j-1}}^{t_{j+1}} b_1^2(x, s)\, ds \leq \alpha^* \right\} \tag{4.45}$$

for α^* small enough and $\sigma_i = 1$ [$\sigma_i = -1$] if i is odd [even] and $\tilde{\xi}_i(t)$ in the i-th mixed contour of $m(\cdot, t)$. From the definition of A_{α^*} we also have that

$$|A^c_{\alpha^*}| \leq \frac{8}{\alpha^*} \int_{t_{j-1}}^{t_{j+1}} \|\alpha b_1(s)\|^2_{L^2(d\nu)} ds, \tag{4.46}$$

where

$$d\nu(x) := \frac{1}{1 - \bar{m}^2_{\tilde{\xi}(t)}} dx.$$

Moreover, we call $\Lambda_i(t)$, $i = 1, .., k$, the open intervals $\frac{1}{2}\big(\tilde{\xi}_{i-1}(t) + \tilde{\xi}_i(t), \tilde{\xi}_{i+1}(t) + \tilde{\xi}_i(t)\big)$, with $\tilde{\xi}_0(t) = -\infty$ and $\tilde{\xi}_{k+1}(t) = +\infty$. We have the following estimate

$$|\tilde{\xi}_i(t) - \xi_i(t)| + \|u(\cdot, t) - \{m(\cdot, t) - \bar{m}_{\bar{\xi}(t)}\}\|_{L^2(d\nu)} \leq \frac{c}{\alpha^*} \int_{t_{j-1}}^{t_{j+1}} \|\alpha b_1(s)\|^2_{L^2(d\nu)} ds. \tag{4.47}$$

In the next proposition we give upper bounds for displacements of centers with i odd and lower bounds for those with i even. In the proof, we follow the strategy in [20] with the exception of having a different operator and therefore we have to work in an appropriately weighted space.

Proposition 4.14. *There is a constant $c_{4.14} > 0$, so that for ϑ and δ small enough and for all $t \in [t_j, t_{j+1}]$, we have the following bounds:*

$$\|u(\cdot,t)\|^2_{L^2(d\nu)} \le e^{-(t-t_j)\omega}\|u(\cdot,t_{\rm in})\|^2_{L^2(d\nu)} + c_{4.14} S U_j^2, \tag{4.48}$$

$$\sigma_i[\xi_i(t) - \xi_i(t_{\rm in})] \le -\frac{1}{\|\bar m'\|_2^2}\int_{t_{\rm in}}^t (\alpha b_1, \bar m'_{\xi_i(t)})_{L^2(d\nu)} + c_{4.14}\big[\|u(\cdot,t_{\rm in})\|^2_{L^2(d\nu)} + U_j^2\big] \tag{4.49}$$

where $i = 1, .., k$ and

$$U_j^2 = \int_{t_j}^{t_{j+1}} \|\alpha b_1\|^2_{L^2(d\nu)} + S R_{\max}, \quad R_{\max} = c_{4.14} e^{-\alpha|\log\epsilon|^2/2}. \tag{4.50}$$

Note that $R_{\max} \to 0$ as $\epsilon \to 0$.

Proof. Let

$$L:\ L^2(\mathbb{R}, d\nu) \to L^2(\mathbb{R}, d\nu),\ (Lu)(x) := -u(x) + (1 - \bar m^2_{\tilde\xi(t)})(\beta J * u)(x),$$

where

$$d\nu(x) := \frac{dx}{1 - \bar m^2_{\tilde\xi(t)}(x)}.$$

For $x \in \Lambda_i$, we have

$$\frac{du(x,t)}{dt} = \sigma_i \dot{\tilde\xi}_i(t)\bar m'_{\tilde\xi_i(t)} + Lu(x,t) + \tilde R(u) + \alpha b_1(x,t), \tag{4.51}$$

where

$$\tilde R(u) := G''\big(\beta J * (\bar m_{\tilde\xi(t)} + (1-\mu_0)\lambda_0 u)\big)\big(\beta J * u\big)^2,$$

with

$$0 \le \lambda_0,\ \mu_0 \le 1$$

and

$$G(x) := \tanh x.$$

It is an easy calculation to show that

$$\|\tilde R(u)\|_{L^1(d\nu)} \le c\|u\|^2_{L^2(d\nu)}. \tag{4.52}$$

By multiplying (4.51) by $u(\cdot,t)\mathbf{1}_{A_{\alpha^*}}$ and integrating over space we obtain:

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\left(\frac12\|u\mathbf{1}_{A_{\alpha^*}}\|^2_{L^2(d\nu)}\right) &= (u\mathbf{1}_{A_{\alpha^*}},\, Lu)_{L^2(d\nu)} + (u\mathbf{1}_{A_{\alpha^*}},\, \tilde R(u))_{L^2(d\nu)} + \\ &\quad + \int_{\mathbb{R}} u\mathbf{1}_{A_{\alpha^*}}\alpha\, b_1 d\nu + R(t),\end{aligned} \tag{4.53}$$

where

$$R(t)=\sum_{i=1}^{k}\sigma_i\dot{\tilde{\xi}}(t)\left(\int_{\Lambda_i}\bar{m}_{\tilde{\xi}_i(t)}u\mathbf{1}_{A_{\alpha^*}}\,d\nu+\int_{\Lambda_i}\bar{m}_{\tilde{\xi}_i(t)}\frac{\bar{m}_{\tilde{\xi}_i(t)}}{1-\bar{m}^2_{\tilde{\xi}_i(t)}}u^2\mathbf{1}_{A_{\alpha^*}}d\nu\right). \tag{4.54}$$

By (4.46),

$$\left|(u\mathbf{1}_{A_{\alpha^*}},Lu)_{L^2(d\nu)}-(u\mathbf{1}_{A_{\alpha^*}},L(u\mathbf{1}_{A_{\alpha^*}}))_{L^2(d\nu)}\right|\le\frac{32}{\alpha^*}\int_{t_{j-1}}^{t_{j+1}}\|\alpha b_1(s)\|^2_{L^2(d\nu)}.$$

By the spectral gap property, $(u\mathbf{1}_{A_{\alpha^*}},Lu\mathbf{1}_{A_{\alpha^*}})_{L^2(d\nu)}\le-\omega\|u\|_{L^2(d\nu)}$ and by using a similar estimate on $\|u\|_{L^\infty}$ as in Theorem C.3 of Appendix in [20] in order to bound the second term in (4.53), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\|^2_{L^2(d\nu)}\right)\le-\omega\|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}+c(\epsilon_1+c_1\|u\|_{L^2(d\nu)})^{2/3}\|u\mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}$$
$$+(u\mathbf{1}_{A_{\alpha^*}},\alpha b_1)_{L^2(d\nu)}+c'\int_{t_j}^{t_{j+1}}\|\alpha b_1(s)\|^2_{L^2(d\nu)}+R(t).$$

Let

$$\tau:=\inf\left\{t:\|u(\cdot,t)\|^{2/3}_{L^2(d\nu)}>\frac{\omega}{8cc_1}\right\}. \tag{4.55}$$

Bounding $\|(u\mathbf{1}_{A_{\alpha^*}},\alpha b_1)\|_{L^2(d\nu)}\le\frac{2\|\alpha b_1\|^2_{L^2(d\nu)}}{\omega}+\frac{\omega\|u\mathbf{1}_{A_{\alpha^*}}\|^2_{L^2(d\nu)}}{4}$, for all times $t\in[t_j,t_{j+1}]$ such that $t<\tau$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\mathbf{1}_{A_{\alpha^*}}\|^2_{L^2(d\nu)}\right)\le-\frac{\omega}{2}\|u\mathbf{1}_{A_{\alpha^*}}\|^2_{L^2(d\nu)}+\frac{2}{\omega}\|\alpha b_1\|^2_{L^2(d\nu)}+R(t),$$

i.e., for $t^*=\min\{\tau,t_{j+1}\}$ we obtain

$$\|\mathbf{1}_{A_{\alpha^*}}u(\cdot,t^*)\|^2_{L^2(d\nu)}\le e^{-(t^*-t_j)\omega}\|u(\cdot,t_j)\|^2_{L^2(d\nu)}+c_{4.14}\left(\int_{t_j}^{t^*}\|\alpha b_1(s)\|^2_{L^2(d\nu)}+SR_{\max}\right),$$

with $R_{\max}$ defined in (4.50). Since

$$\|u\|^2_{L^2(\mathrm{d}\nu)}\le\|\mathbf{1}_{A_{\alpha^*}}u\|^2_{L^2(\mathrm{d}\nu)}+\frac{4}{\alpha^*}\int_{t_j}^{t_{j+1}}\|\alpha b_1(s)\|^2_{L^2(d\nu)},$$

we have

$$\|u(\cdot,t^*)\|^2_{L^2(d\nu)}\le e^{-(t^*-t_j)\omega}\|u(\cdot,t_j)\|^2_{L^2(d\nu)}+c_{4.14}\left(\int_{t_j}^{t^*}\|\alpha b_1(s)\|^2_{L^2(d\nu)}+SR_{\max}\right).$$

By the choice of δ in (4.34) and (4.97) we have

$$c_{4.14}\int_{t_j}^{t^*}\|\alpha b_1(s)\|^2_{L^2(d\nu)}+SR_{\max}\le c_{4.14}\left(\frac{1}{1-c_*^2C\Delta(\epsilon)}\delta+SR_{\max}\right)\le10^{-3}.$$

Thus, for δ, ϑ and ϵ small enough, $\|u(\cdot,t^*)\|^2_{L^2(d\nu)} \le (\frac{\omega}{8cc_1})^3$ and hence $t^* = t_{j+1}$.

For the proof of (4.49), we multiply (4.51) by $\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i(t)}$ and then estimate $(\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i(t)}, u_t)_{L^2(d\nu)}$ by first writing (4.44) as

$$(\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i(t)}, \sigma_i\bar{m}_{\tilde{\xi}_i(t)} - \bar{m}_{\tilde{\xi}(t)})_{L^2(d\nu)} = (\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}(t)}, u)_{L^2(d\nu)}, \tag{4.56}$$

after adding and subtracting $\bar{m}_{\tilde{\xi}(t)}$. Since the measure $d\nu$ depends on time, we also have:

$$\begin{aligned}\frac{d}{dt}(\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i(t)}, u)_{L^2(d\nu)} &= (\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i(t)}, u_t)_{L^2(d\nu)} + (\mathbf{1}_{A_{\alpha^*}}\bar{m}''_{\tilde{\xi}_i}\sigma_i\dot{\tilde{\xi}}_i, u)_{L^2(d\nu)} \\ &\quad + \sum_j \int_{\Lambda_j} \bar{m}'_{\tilde{\xi}_i}\mathbf{1}_{A_{\alpha^*}} u \frac{2\bar{m}_{\tilde{\xi}_j}\bar{m}'_{\tilde{\xi}_j}\dot{\tilde{\xi}}_j}{(1-\bar{m}^2_{\tilde{\xi}_j})^2}dx. \end{aligned}\tag{4.57}$$

We obtain:

$$\begin{aligned}(\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i}, u_t)_{L^2(d\nu)} &= \dot{\tilde{\xi}}_i\left\{(\mathbf{1}_{A_{\alpha^*}}\bar{m}''_{\tilde{\xi}_i}, u)_{L^2(d\nu)} + (\mathbf{1}_{A_{\alpha^*}}\bar{m}''_{\tilde{\xi}_i}, \bar{m}_{\tilde{\xi}} - \sigma_i\bar{m}_{\tilde{\xi}_i})_{L^2(d\nu)}\right\} \\ &\quad - \sum_{j\neq i}\left(\mathbf{1}_{A_{\alpha^*}}\mathbf{1}_{\Lambda_j}\bar{m}'_{\tilde{\xi}_i}, (\sigma_i\dot{\tilde{\xi}}_i\bar{m}'_{\tilde{\xi}_i} - \sigma_j\dot{\tilde{\xi}}_j\bar{m}'_{\tilde{\xi}_j})\right)_{L^2(d\nu)} \\ &\quad + \sum_{j\neq i}\int_{\Lambda_j} 2\mathbf{1}_{A_{\alpha^*}} u\bar{m}_{\tilde{\xi}_j}\frac{\bar{m}'_{\tilde{\xi}_i}\bar{m}'_{\tilde{\xi}_j}}{1-\bar{m}^2_{\tilde{\xi}_j}}d\nu \\ &\quad - \sum_{j\neq i}\int_{\Lambda_j}\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i}(\sigma_i\bar{m}_{\tilde{\xi}_i} - \bar{m}_{\tilde{\xi}})\frac{1}{1-\bar{m}^2_{\tilde{\xi}_j}}2\bar{m}_{\tilde{\xi}_j}\bar{m}'_{\tilde{\xi}_j}\sigma_j\dot{\tilde{\xi}}_j d\nu. \end{aligned}\tag{4.58}$$

On the other hand, in (4.51) we have:

$$(\mathbf{1}_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i}, Lu)_{L^2(d\nu)} = (u, L\bar{m}'_{\tilde{\xi}_i})_{L^2(d\nu)}, \quad \text{with } |L\bar{m}'_{\tilde{\xi}_i}| \le R_{\max}.$$

Thus, from (4.51) and (4.58) we obtain:

$$\begin{aligned}&\sigma_i\dot{\tilde{\xi}}_i\left[\|\bar{m}'_{\tilde{\xi}_i}\mathbf{1}_{A_{\alpha^*}}\|^2_{L^2(d\nu)} - \sigma_i\{(\mathbf{1}_{A_{\alpha^*}}\bar{m}''_{\tilde{\xi}_i}, u)_{L^2(d\nu)} + (\mathbf{1}_{A_{\alpha^*}}\bar{m}''_{\tilde{\xi}_i}, \bar{m}_{\tilde{\xi}} - \sigma_i\bar{m}_{\tilde{\xi}_i})_{L^2(d\nu)}\}\right] \\ &+ \sum_{j\neq i}\left(\mathbf{1}_{A_{\alpha^*}}\mathbf{1}_{\Lambda_j}\bar{m}'_{\tilde{\xi}_i}, (\sigma_i\dot{\tilde{\xi}}_i\bar{m}'_{\tilde{\xi}_i} - \sigma_j\dot{\tilde{\xi}}_j\bar{m}'_{\tilde{\xi}_j})\right)_{L^2(d\nu)} - \sum_{j\neq i}\int_{\Lambda_j} 1_{A_{\alpha^*}} 2u\bar{m}_{\tilde{\xi}_j}\frac{\bar{m}'_{\tilde{\xi}_i}\bar{m}'_{\tilde{\xi}_j}}{1-\bar{m}^2_{\tilde{\xi}_j}}d\nu \\ &\qquad + \sum_{j\neq i}\int_{\Lambda_j} 1_{A_{\alpha^*}}\bar{m}'_{\tilde{\xi}_i}(\sigma_i\bar{m}_{\tilde{\xi}_i} - \bar{m}_{\tilde{\xi}})\frac{1}{1-\bar{m}^2_{\tilde{\xi}_j}}2\bar{m}_{\tilde{\xi}_j}\bar{m}'_{\tilde{\xi}_j}\sigma_j\dot{\tilde{\xi}}_j d\nu \\ &\qquad \le -(\bar{m}'_{\tilde{\xi}_i}, \alpha b_1)_{L^2(d\nu)} + |A^c_{\alpha^*}| + c'c\|\mathbf{1}_{A_{\alpha^*}}u\|^2_{L^2(d\nu)} + R_{\max}\end{aligned}$$

which has the form:

$$\sigma_i\|\bar{m}'\|^2_{L^2(d\nu)}\dot{\tilde{\xi}}_i(t) \le \beta_i + \sum_{j=1}^k a_{i,j}|\dot{\tilde{\xi}}_j|, \tag{4.59}$$

where

$$\beta_i = (\mathbf{1}_{A_{\alpha^*}}\bar m''_{\tilde\xi_i}, u)_{L^2(d\nu)} - (\bar m'_{\tilde\xi_i}, \alpha b_1)_{L^2(d\nu)} + c'c\|u\|^2_{L^2(d\nu)} + |A^c_{\alpha^*}| + R_{\max}, \tag{4.60}$$

with

$$\begin{aligned}|\beta_i + (\bar m'_{\tilde\xi_i}, \alpha b_1)_{L^2(d\nu)}| &\le c''[e^{-(t-t_{\rm in})\omega}\|u(\cdot, t_{\rm in})\|^2_{L^2(d\nu)} \\ &+ \int_{t_{\rm in}}^t \|\alpha b_1\|^2_{L^2(d\nu)} ds + SR_{\max} + \|1 - \mathbf{1}_{A_{\alpha^*}}\|_{L^2(d\nu)}\|\alpha b_1\|_{L^2(d\nu)}\end{aligned}$$

and

$$\begin{aligned} a_{i,j} &= (\mathbf{1}_{\Lambda_j}\bar m''_{\tilde\xi_j}, \bar m_{\tilde\xi_i} - \sigma_j \bar m_{\tilde\xi_j}) + (\mathbf{1}_{\Lambda_j}\bar m'_{\tilde\xi_i}, \bar m'_{\tilde\xi_j})_{L^2(d\nu)} \\ &+ \int_{\Lambda_j} 2u\bar m_{\tilde\xi_j}\frac{\bar m'_{\tilde\xi_i}\bar m'_{\tilde\xi_j}}{1-\bar m^2_{\tilde\xi_j}} d\nu - \int_{\Lambda_j} \bar m'_{\tilde\xi_i}(\sigma_i \bar m_{\tilde\xi_i} - \bar m_{\tilde\xi})\frac{2\bar m_{\tilde\xi_j}\bar m'_{\tilde\xi_j}}{1-\bar m^2_{\tilde\xi_j}} d\nu. \end{aligned} \tag{4.61}$$

Then we conclude the proof in the same fashion as in [20] by estimating $a_{i,j}$, since ξ_i and ξ_j are well separated. □

Concluding this section, we recall that we constructed $m(t, \cdot)$ for $t \in [t_j, t_{j+1}]$ and obtained estimates for the error $\|m(\cdot, t) - \bar m_{\bar\xi(t)}\|^2_{L^2(d\nu)}$. Next we define $m(\cdot, t^+_{j+1})$ in order to apply this linearization procedure in the whole of the maximal connected component G.

4.3.6 From a good time interval to the next

The result of Proposition 4.14 ensures that during the good time interval $[t_j, t_{j+1})$ the solution of (4.42) is close to a moving instanton. More precisely, by (4.34) we have that $c_{4.14}U_j^2 \le \vartheta$ and by (4.33) that $e^{-\omega S} \le 1/2$. Then by (4.48) we get, supposing ϵ small enough,

$$\|u(\cdot, t_{j+1})\|^2_{L^2(d\nu)} \le e^{-\omega S}\|u(t_j)\|^2_{L^2(d\nu)} + c_{4.14}U_j^2 \le 4\vartheta. \tag{4.62}$$

Furthermore, since $\xi_{i+1}(t_{j+1}) - \xi_i(t_{j+1}) \ge |\log\epsilon|^2/2$, as we have seen in the course of the proof of Proposition 4.14, it follows from (4.15) that for ϵ small enough,

$$d_{\mathcal M}(m(\cdot, t_{j+1})) \le 5\vartheta. \tag{4.63}$$

We introduce the notion of *velocity of a front* $\bar m_{\xi_i}(t)$, by defining:

$$v_i^0(t) := \sigma_i \frac{1}{\|\bar m'\|^2_{L^2(d\nu)}}\left|(\alpha b_1, \bar m'_{\xi_i(t)})_{L^2(d\nu)}\right|, \tag{4.64}$$

where again $\sigma_i = 1$ [$\sigma_i = -1$] if i is odd [even]. Moreover, we want to control the position of the centers of $m(\cdot,t)$, so we denote by $r_i(t)$ the leftmost [rightmost] position of the center ξ_i of $m(\cdot,t)$, for i odd [even], taking into account the error in determining the position ξ_i. Thus, the position $r_i(t)$ will be given by ξ_i plus the integral of the velocity induced by the error $\|m(\cdot,t) - \bar{m}_{\bar{\xi}(t)}\|^2_{L^2(d\nu)}$. We define:

$$v_i(t) := v_i^0(t) + \sigma_i c_{4.14}\Big(U_j^2 + \|u(\cdot,t_j)\|^2_{L^2(d\nu)}\Big), \tag{4.65}$$

$$r_i(t) := \xi_i(t_j) + \int_{t_j}^t v_i(s), \qquad \bar{r}(t) = \big(r_1(t), .., r_k(t)\big). \tag{4.66}$$

Notice that $\bar{r}(t) \leq \bar{\xi}(t)$ for $t \in [t_j, t_{j+1})$, where the partial order is defined as:

$$(\xi_1, ..., \xi_k) \geq (\xi_1', ..., \xi_{k'}') \;\Leftrightarrow\; \bar{m}_{(\xi_1,...,\xi_k)} \geq \bar{m}_{(\xi_1',...,\xi_{k'}')}. \tag{4.67}$$

In particular, if $k = k'$,

$$(\xi_1, ..., \xi_k) \geq (\xi_1', ..., \xi_k') \;\Leftrightarrow\; \xi_i \leq \xi_i', \text{i odd}, \quad \xi_i \geq \xi_i', \text{ i even}. \tag{4.68}$$

By the definition of t_{j+1} we know that $d_{\mathcal{M}}(\phi(\cdot,t_{j+1})) \leq \vartheta$. Suppose now that, for $\epsilon > 0$ small enough, $\phi(\cdot,t_{j+1})$ has k'-many mixed contours $\{\tilde{\Gamma}_i\}_{i=1,...,k'}$, k' odd, with $\|\mathbf{1}_{\tilde{\Gamma}_i}(\phi - \bar{m}_{\tilde{\xi}_i})\|_{L^2} \leq \vartheta$ for some $\tilde{\xi}_i \in \tilde{\Gamma}_i$, $i = 1, \ldots, k'$. Note that in general $k' \neq k$ (since m has been re-initialized at t_j and some fronts might have been cancelled). Then by Theorem 4.3 we have that there exist unique centers $\{\xi_i(\phi)(t_{j+1})\}_{i=1,...,k}$ of $\phi(\cdot,t_{j+1})$. The strategy goes as follows: note that since (using (4.41))

$$|b(\phi_1)| = |\alpha b_1| = \left|\left(\frac{1-\bar{m}^2_{\tilde{\xi}(t)}}{8}\right)^{1/2} b_1\right| \leq |b_1| \leq |b(\phi)|,$$

the profile $\phi_1(t_{j+1})$ is expected to have its odd [even] indexed centers on the left [right] of the corresponding centers of $\phi(t_{j+1})$. On the other hand, the profile $m(t_{j+1})$, being a sub-solution of the equation $b(m) = b(\phi_1)$, with initial condition $m(t_j)$ re-initialized as before, it has its odd [even] centers on the right [left] of the corresponding centers of $\phi_1(t_{j+1})$. However, it is not guaranteed that this is also the case with the centers of $\phi(t_{j+1})$. Therefore, since in the next good time interval we choose $\phi_1(\cdot,t^+_{j+1}) := \phi(\cdot,t^+_{j+1})$ we need to re-initialize $m(\cdot,t^+_{j+1})$ to be such that $m(\cdot,t^+_{j+1}) \leq \phi_1(\cdot,t^+_{j+1})$ and keep track of the relevant error. As a result of the initialization, the profile $m(t_{j+1})$ may have fewer centers than $\phi_1(\cdot,t_{j+1})$.

We estimate the distance between the corresponding centers of ϕ and m at t_{j+1}, when both are close to the manifold $\mathcal{M}$. Recall also that, by the initialization, the centers at t_j^+

have mutual distance $\geq |\log \epsilon|^2$. To perform our estimate we introduce an auxiliary profile ϕ_2 by putting as forcing term only b_1 with the same initial condition. For $t \in [t_j, t_{j+1})$ we have:

$$\|\phi(t)-\phi_2(t)\|_{L^1} \leq \int_{t_j}^{t} e^{-(t-s+t_j)} \beta \|J\|_{L^1} \|\phi(s)-\phi_2(s)\|_{L^1} ds + \int_{t_j}^{t} \int_{\mathbb{R}} e^{-(t-s+t_j)} |b-b_1| dx\, ds,$$

where

$$\int_{t_j}^{t} \int_{\mathbb{R}} e^{-(t-s+t_j)} |b-b_1| dx\, ds \leq \int_{|b|>\Delta(\epsilon)\}} |b| dx\, ds.$$

In the good time interval $[t_j, t_{j+1})$ we define the quantity:

$$\delta_j := \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(b,u,w)(x,s) ds\, dx, \tag{4.69}$$

in which case it is of the order $\delta(\epsilon)$. From (3.24) we obtain that:

$$\begin{aligned}
\delta_j = \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(b,u,w)(x,s) ds\, dx &\geq \int_{\{|b|>\Delta(\epsilon)\}} \mathcal{H}(b,u,w)(x,s) ds\, dx \\
&\geq C \int_{\{|b|>\Delta(\epsilon)\}} |b| \log(|b|+1) ds\, dx \\
&\geq C \int_{\{|b|>\Delta(\epsilon)\}} |b| \log(1+\Delta(\epsilon)) ds\, dx.
\end{aligned}$$

Thus, (since $\|J\|_{L^1} = 1$)

$$\|\phi(\cdot,t)-\phi_2(\cdot,t)\|_{L^1} \leq \beta \int_{t_j}^{t} e^{-(t-s+t_j)} \|\phi(\cdot,s)-\phi_2(\cdot,s)\|_{L^1} ds + \frac{\delta_j}{C\log(1+\Delta(\epsilon))} \tag{4.70}$$

and for a new constant $C > 0$ by Gronwall's lemma we obtain that

$$\|\phi(\cdot,t_{j+1}) - \phi_2(\cdot,t_{j+1})\|_{L^1} \leq C e^{(2+\beta)S} \frac{\delta_j}{\Delta(\epsilon)}. \tag{4.71}$$

On the other hand, comparing to m we have

$$\frac{d}{dt} \int_{\mathbb{R}} (\phi_2 - m)^2(x,t)\, dx =$$

$$\begin{aligned}
&= -2 \int_{\mathbb{R}} (\phi_2 - m)^2(x,t)\, dx + 2 \int_{\mathbb{R}} (1-\alpha) b_1(x,t)(\phi_2 - m)(x,t)\, dx \\
&\quad +2 \int_{\mathbb{R}} (\phi_2 - m)(x,t)(\tanh(\beta J * \phi_2(x,t)) - \tanh(\beta J * m^0(x,t))) dx \\
&\leq C \int_{\mathbb{R}} (\phi_2 - m)^2(x,t)\, dx + c \int_{\mathbb{R}} (1-\alpha)^2 b_1^2(x,t)\, dx.
\end{aligned}$$

Since from (4.41) it holds that $1-\alpha \le (c^*-1)\alpha$, applying Gronwall's inequality and using (4.97) which is given later in Sect. 4.7, we obtain

$$\begin{aligned} \|\phi_2(\cdot,t)-m(\cdot,t)\|^2_{L^2} &\le ce^{(2+\beta)(t-t_j)}\int_{\mathbb{R}}\int_{t_j}^t \alpha^2 b_1^2 ds\, dx \\ &\le ce^{(2+\beta)S}\frac{1}{1-c_*^2 C\Delta(\epsilon)}\,\delta_j, \end{aligned} \tag{4.72}$$

for ϵ small enough so that $c_*^2 C\Delta(\epsilon)<1$. Thus, since $\|\phi(\cdot,t)-\phi_2(\cdot,t)\|_{L^\infty(\mathbb{R})}\le 2$, (4.70) and (4.72) yield

$$\begin{aligned} \|\phi(\cdot,t)-m(\cdot,t)\|^2_{L^2(\mathbb{R})} &\le 2\|\phi_2(\cdot,t)-m(\cdot,t)\|^2_{L^2(\mathbb{R})}(t)+4\|\phi(\cdot,t)-\phi_2(\cdot,t)\|_{L^1(\mathbb{R})} \\ &\le C\frac{\delta_j}{\Delta(\epsilon)}+ce^{(2+\beta)S}\frac{1}{1-c_*^2 C\Delta(\epsilon)}\,\delta_j =: S^j_\epsilon, \end{aligned} \tag{4.73}$$

where by choosing $\kappa<\lambda$ in the definition of $\Delta(\epsilon)$ in (4.36), we have that $S^j_\epsilon \to 0$ as $\epsilon\to 0$. Using the above estimate and the fact that both m and ϕ are close to the manifold at time t_j, we obtain that

$$|\xi(\phi)(t_{j+1})-\xi(m)(t_{j+1})| \le \|\bar m_{\bar\xi(m)}-\bar m_{\bar\xi(\phi)}\| \le S^j_\epsilon + 6\vartheta. \tag{4.74}$$

Next, recalling the definition of $\bar r(t)$ in (4.66), in order to define $r_i(t^+_{j+1})$ we consider the quantity

$$\hat r_i(t_{j+1}) := r_i(t_{j+1})+\sigma_i S^j_\epsilon \tag{4.75}$$

and we erase all pairs i, $i+1$ such that $\hat r_{i+1}(t_{j+1})-\hat r_i(t_{j+1})\le |\log\epsilon|^2$. Then we let

$$r_i(t^+_{j+1}) := \hat r_i(t_{j+1}),$$

if no such erasing has occurred for the index i. Otherwise, we let $r_i(t^+_{j+1}) := \emptyset$.

In Section 4.4 we introduce the notion of particles while referring to the fronts and we say that in this case the particles i and $i+1$ have collided and, due to this collision, they disappeared. We will also write that $r_i(t)=r_{i+1}(t)=\emptyset$ for $t>t_{j+1}$. Moreover, note that the function $\bar r(t)$ has jumps at the times between good time intervals and this fact will be taken into account in the estimation of the total displacement and the corresponding "macroscopic" cost expressed in terms of the cost due to the motion of the particles. For the re-initialization at t^+_{j+1} we define:

$$m(\cdot,t^+_{j+1}) := \min\{\phi(\cdot,t_{j+1}),\, \bar m_{r_i(t^+_{j+1})}(\cdot)\}. \tag{4.76}$$

In this way we ensure that $m(\cdot, t_{j+1}^+) \le \phi(\cdot, t_{j+1})$ as well as that $r_i(t_{j+1}^+)$ is a lower [upper] bound of $\xi_i(m(\cdot, t_{j+1}^+))$ for i odd [even]. Thus, taking ϵ small enough we have that $d_{\mathcal{M}}(m(\cdot, t_{j+1}^+)) \le 20\vartheta$ and that its centers have mutual distance $\ge |\log \epsilon|^2$. So we can repeat the same procedure for the next good time interval $[t_{j+1},\, t_{j+2})$.

4.3.7 Displacement during the bad time intervals

From (4.35) the maximal length of the connected component of bad time intervals is bounded by $S\frac{P}{\delta(\epsilon)} << |\log \epsilon|^2$ for the choice of $\delta(\epsilon)$ made in (4.36). Moreover, the applied force b can be related to and bounded by the cost. Therefore, the displacement of the already existing centers should be smaller than $|\log \epsilon|^2$, which is the distance between the appropriately initialized centers of the interfaces. Similarly, the newly nucleated fronts are also at a distance from each other smaller than $|\log \epsilon|^2$ even at the end of the connected component of the bad time intervals. Hence, overall the motion during the bad time intervals will be negligible macroscopically.

Suppose that $[t_{j'}, t_{j''})$ is a connected component of bad time intervals. Recalling the construction of the partition of good and bad time intervals in subsection 4.3.3, we have that $t_k = kS$, for all $j' \le k \le j''$, $k \in \mathbb{N}$. In the connected component of bad time intervals we define the profile m by solving the equation

$$\frac{d}{dt} m = -m + \tanh(\beta J * m) + b(\phi), \tag{4.77}$$

with initial condition the profile $m(t_{j'}, \cdot)$ as we obtained it from the previous good time interval. Invoking again Corollary 4.6 and the choice of S for the profile m constructed above, for $j'+1 \le k \le j''$ there exist $\bar{t}_k \in [t_j, t_{j+1})$ with $m(\bar{t}_k, \cdot)$ close to $\mathcal{M}$.

We compare the solution m to the solution m^0 of the same equation without the forcing term $b(\phi)$ for the interval $[t_{j'}, \bar{t}_{j'+1})$, both with the same initial condition. To do that we compare both of them to the auxiliary profile ϕ_2 generated by the force b_1. From (4.71), we have that

$$\|m(\cdot, \bar{t}_{j'+1}) - \phi_2(\cdot, \bar{t}_{j'+1})\|_{L^2}^2 \le e^{(2+\beta)S} \frac{\delta_{j'}}{\Delta(\epsilon)}. \tag{4.78}$$

Similarly to (4.72) we have:

$$\frac{d}{dt} \int_{\mathbb{R}} (\phi_2 - m^0)^2(x,t)\, dx =$$

$$\begin{aligned}
&= -2\int_{\mathbb{R}}(\phi_2-m^0)^2(x,t)\,dx+2\int_{\mathbb{R}}b_1(x,t)(\phi_2-m^0)(x,t)\,dx\\
&\quad +2\int_{\mathbb{R}}(\phi_2-m^0)(x,t)(\tanh(\beta J*\phi_2(x,t))-\tanh(\beta J*m^0(x,t)))dx\\
&\le C\int_{\mathbb{R}}(\phi_2-m^0)^2(x,t)\,dx+c\int_{\mathbb{R}}\alpha^2 b_1^2(x,t)\,dx,
\end{aligned}$$

for c large enough. After applying Gronwall's inequality and (4.97) (given later in Sect. 4.7), we obtain:

$$\|\phi_2(\cdot,\bar{t}_{j'+1})-m^0(\cdot,\bar{t}_{j'+1})\|^2_{L^2}\le ce^{(2+\beta)S}\frac{1}{1-c_*^2C\Delta(\epsilon)}\,\delta_{j'},\tag{4.79}$$

where $\delta_{j'}$ has been defined in (4.69). Combining (4.78) and (4.79), for m constructed in (4.77) we have:

$$\|m(\cdot,\bar{t}_{j'+1})-m^0(\cdot,\bar{t}_{j'+1})\|^2_{L^2(\mathbb{R})}\le ce^{(2+\beta)S}\frac{\delta_{j'}}{\Delta(\epsilon)}.\tag{4.80}$$

Moreover, since by the definition of the time $\bar{t}_{j'+1}$ the profile m is close to $\mathcal{M}$ at that time, we have that

$$\|\bar{m}_{\bar{\xi}(m(\cdot,\bar{t}_{j'+1}))}-\bar{m}_{\bar{\xi}(m^0(\cdot,\bar{t}_{j'+1}))}\|^2_{L^2(\mathbb{R})}\le ce^{(2+\beta)S}\frac{\delta_{j'}}{\Delta(\epsilon)}+7\vartheta,\tag{4.81}$$

for some $c>0$. From this, we can obtain an estimate for the distance between the centers in $\bar{\xi}(m(\cdot,\bar{t}_{j'+1}))$ and $\bar{\xi}(m^0(\cdot,\bar{t}_{j'+1}))$. Let k be the number of centers of $m(\cdot,t_{j'})$ and $\bar{r}(t_{j'})=(r_1(t_{j'}),...,r_k(t_{j'}))$ with $|r_{i+1}(t_{j'})-r_i(t_{j'})|\ge|\log\epsilon|^2$, $\forall i$. For $l\in\{1,\dots,k\}$ odd, define i_l to be the odd label such that

$$\min_{i\,\mathrm{odd}}|\xi_i-\xi_l^0|=|\xi_{i_l}-\xi_l^0|.\tag{4.82}$$

For l even we define i_l analogously. Furthermore, during the time interval $[t_{j'},\bar{t}_{j'+1})$, new centers might be created due to nucleations. Let $\ell_1,\dots,\ell_p$ be the labels of the newly created centers.

By the properties of the instanton we have that the upper bound in (4.81) induces an upper bound on the volume of the mismatch between $\bar{m}_{\bar{\xi}(\phi(\cdot,\bar{t}_{j'+1}))}$ and $\bar{m}_{\bar{\xi}(m^0(\cdot,\bar{t}_{j'+1}))}$. Since the centers $i_1,\dots,i_k$ are still far enough, this further induces a bound on the corresponding centers. Hence, both $|\xi_{i_l}-r_l|$ and $|\xi_{\ell_i}-\xi_{\ell_{i+1}}|$, for i odd in $\{1,\dots,k\}$ are bounded by the estimate in (4.81).

In the next iteration, we construct a profile solving (4.77) for $t\ge\bar{t}_{j'+1}$ starting at $m(\bar{t}_{j'+1},\cdot)$. Using the same argument as before, we choose another time $\bar{t}_{j'+2}\in[j'+2-$

$\frac{1}{2}, j'+2-\frac{1}{4}]S$ with $m(\bar t_{j'+2},\cdot)$ close to $\mathcal{M}$. By repeating the same procedure we obtain

$$\|m(\cdot,\bar t_{j'+2}) - m^0(\cdot,\bar t_{j'+2})\|^2_{L^2(\mathbb{R})} \le ce^{(2+\beta)S}\frac{\delta_{j'+1}}{\Delta(\epsilon)}, \tag{4.83}$$

where m^0 is the solution of the equation without the forcing term in the interval $[\bar t_{j'+1},\bar t_{j'+2})$ starting at $m(\cdot,\bar t_{j'+1})$. This induces a bound on the corresponding centers by the same amount. These could be the original ones, or the ones nucleated in the time interval $[t_{j'},\bar t_{j'+1})$ and continued moving the current one, or those nucleated during the second time interval $[\bar t_{j'+1},\bar t_{j'+2})$. Thus, during the first two bad time intervals of the connected component $[t_{j'},t_{j''})$, the displacement of the old centers (at time $t_{j'}$) or the distance between the newly created are both bounded by

$$ce^{(2+\beta)S}\frac{\delta_{j'}}{\Delta(\epsilon)} + 7\vartheta + ce^{(2+\beta)S}\frac{\delta_{j'+1}}{\Delta(\epsilon)} + 7\vartheta.$$

At the end of the connected component of the bad time intervals the corresponding estimate is

$$ce^{(2+\beta)S}\frac{1}{\Delta(\epsilon)}\sum_{k=j'}^{j''}\delta_k + \frac{P}{\delta(\epsilon)}7\vartheta \le ce^{(2+\beta)S}\frac{P}{\Delta(\epsilon)} + \frac{P}{\delta(\epsilon)}7\vartheta << |\log\epsilon|^2, \tag{4.84}$$

by the choice in (4.36).

4.4 The particle model, total cost and total displacement

4.4.1 The "particle" model

Given a profile $\phi \in \mathcal{U}[\epsilon^{-1}R,\epsilon^{-2}T]$, in the previous sections we created a function m with $I(\phi) \ge I(m)$. By construction, see (4.76), at the end of each good time interval the function m has its odd/even centers on the right/left of the corresponding centers of ϕ, eventually after performing a jump by a quantity S_ϵ (see (4.73)), if necessary. To each such center we assign a "particle" whose position is given by the function $t \mapsto r_i(t)$ as defined in (4.66). From (4.27) there is a maximum possible number of such particles, say n^* and we write $\bar r(t) := (r_1(t),\dots,r_{n^*}(t))$ for their positions. During a connected component of good time intervals we may have that some of these particles die as a result of a "collision" as described before. On the other hand, during the bad time intervals (where the cost is higher) we may get a birth (or more) of two such particles after the occurrence of a nucleation. Thus, a possible behavior of these particles is the following:

at time $t = 0$ we have the particle $r_1(0) = 0$ and $r_i(0) = \emptyset$ for all $2 \leq i \leq n$, which moves in a bad time interval, during which a nucleation takes place at time $t_1^* \geq 0$ and we have the creation of the new particles at positions $r_{i_1}(t_1^*) = r_{i_1+1}(t_1^*)$ (distance $|\log\epsilon|^2$), with i_1 odd (note also that we let $r_{i_1}(t) = r_{i_1+1}(t) = \emptyset$ for $t < t_1^*$). Then the particles enter into a connected component of good time intervals after (possibly) making a jump in their positions r_i by at most $o(|\log\epsilon|^2)$ as shown in Section 4.3.7. Then, before entering into the next good time intervals of small cost, new jumps may occur as a result of the initialization described in Section 4.3.1. After entering, new jumps have to be taken into account as a result of a jump from a good time interval to the next as in Section 4.3.6. In both of these cases (say at a time t_2^*) it may happen that two particles (r_{i_2} and r_{i_2+1}) collapse in which case we write $r_{i_2}(t) = r_{i_2+1}(t) = \emptyset$ for all $t \geq t_2^*$. Hence, following the above rules and the analysis in the previous sections we obtain the configuration of the particles denoted by $\{n, (r_1(t), \dots, r_n(t))\}$ for $t \in [0, \epsilon^{-2}T]$.

4.4.2 Lower bound

We want to find a lower bound of the total cost determined by the new quantities $\bar{r}(t)$ and the velocities $v_i^0(t)$. Furthermore, we have the constraint that the total displacement is $\geq \epsilon^{-1}R$. From this, we derive a constraint on $v_i^0(t)$, for $t \in [0, \epsilon^{-2}T]$. We have to take into account the displacement during the good time intervals, the jumps S_ϵ^j, (4.73), between two good time intervals, the displacement during bad time intervals (4.84) and finally the displacement due to nucleation and collision of particles. Thus, the constraint reads:

$$\begin{aligned} \sum_{i=1}^{n^*} \int_{\{t:\, r_i(t) \neq \emptyset\}} |v_i^0(t)| \;\; &\geq \;\; \epsilon^{-1}R - \Bigg(cn^* \sum_{j \in G_{\text{tot}}} \int_{t_j}^{t_{j+1}} (\|\alpha b_1\|^2_{L^2(d\nu)} + R_{\max}) ds \\ &\qquad + c \sum_{j \in G_{\text{tot}}} S_\epsilon^j + |\log\epsilon|^2 + n^* 4 |\log\epsilon|^2 \Bigg). \end{aligned} \tag{4.85}$$

In the good time interval $[t_j, t_{j+1}]$, using (4.98), we have the following lower bound for the cost:

$$\int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(\phi, \dot{\phi})(x,t)\, dx\, dt \geq \int_{t_j}^{t_{j+1}} \|\alpha b_1\|_{L^2(d\nu)}\, dt - \frac{c_*^2 C \Delta(\epsilon)}{1 - c_*^2 C \Delta(\epsilon)} P,$$

where by Hölder's inequality we also have that

$$\|\alpha b_1\|_{L^2(d\nu)} \geq \sum_{i:\, r_i(t) \neq \emptyset} \left\{ \frac{1}{\|\bar{m}'\|^2_{L^2(d\nu)}} \left| (\alpha b_1, \bar{m}'_{\xi_i(t)})_{L^2(d\nu)} \right| - ce^{-\alpha |\log\epsilon|^2/2} \right\}.$$

Thus, taking also into account the mobility $\mu = 4\|\bar{m}'\|_{L^2(d\nu)}$, in a good time interval we obtain:

$$\int_{t_j}^{t_{j+1}} \int_{\mathbb{R}} \mathcal{H}(\phi,\dot{\phi})(x,t)\,dx\,dt \geq \int_{t_j}^{t_{j+1}} \sum_{i:r_i(t)\neq\emptyset} \frac{v_i^0(t)^2}{\mu} - ce^{-\alpha|\log\epsilon|^2/2}2S - \frac{c_*^2 C\Delta(\epsilon)}{1-c_*^2 C\Delta(\epsilon)}P.$$

On the other hand, the cost in a connected component of bad time intervals is neglected unless if a nucleation occurs. Following the notation we used in Section 4.3.7, $[t_{j'}, t_{j''})$ is a generic connected component of bad time intervals. By using the reversibility property (4.4) we have that:

$$\int_{t_{j'}}^{t_{j''}} \int_{\mathbb{R}} \mathcal{H}(\phi,\dot{\phi})(x,t)\,dx\,dt \geq \mathcal{F}(\phi(\cdot,t_{j''})) - \mathcal{F}(\phi(\cdot,t_{j'})).$$

Using (4.29) we have that for the given $\gamma > 0$,

$$\mathcal{F}(\phi(\cdot,t_{j''})) - \mathcal{F}(\phi(\cdot,t_{j'})) \geq 2q\mathcal{F}(\bar{m}) - n^*\gamma,$$

where q is the number of nucleations that happened during $[t_{j'}, t_{j''}]$. Thus, for all $\epsilon > 0$, the total cost is bounded from below by

$$\begin{aligned}\int_0^{\epsilon^{-2}T} \int_{\mathbb{R}} \mathcal{H}(\phi,\dot{\phi})(x,t)\,dx\,dt \quad \geq \quad & \int_{G_{\text{tot}}} \sum_{i:r_i(t)\neq\emptyset} \frac{v_i^0(t)^2}{\mu} + n\mathcal{F}(\bar{m}) - \frac{c_*^2 C\Delta(\epsilon)}{1-c_*^2 C\Delta(\epsilon)}P \\ & -ce^{-\alpha|\log\epsilon|^2/2}\epsilon^{-2}T - \gamma, \end{aligned} \tag{4.86}$$

where $n/2$ is the total number of nucleations with q, $n \leq n^*$ where n^* is the maximum number of fronts created by the nucleations (see (4.27)). Thus, the problem reduces to finding the infimum over the velocities $v_i^0(\cdot)$ of the right hand side of (4.86) under the constraint (4.85), where $i = 1,\dots,n^*$ is the index of a front and suppose that its lifetime is T_i. With this estimate, arguing as in [20] we conclude the proof of the lower bound.

4.4.3 Upper bound

First, we compute the optimal number of nucleations. Then, we construct a sequence $\phi_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$, which at time $t = 0$ consists of a multi-instanton with $2n+1$ centers at positions 0 and $\frac{2i}{2n+1}\epsilon^{-1}R \pm \frac{1}{2}|\log\epsilon|^2$, for $i = 1,\dots,n$. Then for $t \in (0, \epsilon^{-2}T]$ they move with constant velocity $\frac{V}{2n+1}$ to the right (the odd-numbered) or left (the even-numbered), where $V = R/T$. When they are at a distance smaller than $|\log\epsilon|^2$ they disappear. It is easy to check that this sequence satisfies (3.39).

4.5 Existence of solutions of the system (4.38)-(4.40)

Recalling the definition of b in (3.18) and of b_1 in (4.37), we define the sequence $\{\tilde{\xi}^k, \phi_1^k, m^k\}_{k\geq 1}$ which solves the following system of equations (for simplicity we work in the good time interval $[0, S]$):

$$b(\phi_1^k) = \alpha_k b_1, \quad \text{with} \quad \phi_1^k(\cdot, 0) = \phi(\cdot, 0) \quad \text{and} \tag{4.87}$$

$$b(m^k) = b(\phi_1^k), \quad \text{with} \quad m^k(\cdot, 0) = m_0(\cdot), \tag{4.88}$$

where

$$\alpha_0 = 1, \quad \alpha_1 = \left(\frac{1 - \bar{m}^2_{\tilde{\xi}^0}}{8}\right)^{\frac{1}{2}} \quad \text{and} \quad \alpha_k = \left(\frac{1 - \bar{m}^2_{\tilde{\xi}^{k-1}}}{8}\right)^{\frac{1}{2}}.$$

The initial condition m_0 is as in the initialization in Section 4.3.1 and $\tilde{\xi}^k = (\tilde{\xi}_1^k, \ldots, \tilde{\xi}_n^k)$ are the approximate centers of m^k defined as in (4.44). We define the initial center $\tilde{\xi}^0$ as the center of the profile m^0, defined by:

$$b(m^0) = b_1, \quad \text{with} \quad m^0(\cdot, 0) = m_0(\cdot).$$

Then, m^1 solves the following initial value problem:

$$b(m^1) = \alpha_1 b_1, \quad \text{with} \quad m^1(\cdot, 0) = m_0(\cdot).$$

From the equations above for m^0 and m^1 we have:

$$\frac{d}{dt}\|m^1(\cdot, t) - m^0(\cdot, t)\|^2_{L^2} \leq (2 + \beta)\|m^1(\cdot, t) - m^0(\cdot, t)\|^2_{L^2} + \|(1 - \alpha_1)b_1\|^2_{L^2}$$

But, by the definition of c_* in (4.41), it holds that $|(1-\alpha_k)b_1| \leq c_*\alpha_k|b_1|$, for every $k \geq 1$. Then, applying Gronwall's inequality and using (4.97) we obtain:

$$\begin{aligned} \|m^1(\cdot, t) - m^0(\cdot, t)\|^2_{L^2} \quad &\leq \quad ce^{(2+\beta)S}\frac{1}{1 - c_*^2 C\Delta(\epsilon)} \\ &\int_{\mathbb{R}}\int_0^t \mathcal{H}(x, s)ds\,dx \leq ce^{(2+\beta)S}\frac{1}{1 - c_*^2 C\Delta(\epsilon)}\delta(\epsilon), \end{aligned} \tag{4.89}$$

for some new constant $c > 0$. We define

$$\|\tilde{\xi}^k - \tilde{\xi}^{k-1}\| := \max_{i=1,\ldots,n} |\tilde{\xi}_i^k - \tilde{\xi}_i^{k-1}| \tag{4.90}$$

and estimate $|\tilde{\xi}_i^1 - \tilde{\xi}_i^0|$, for $i \in \{1, \ldots, n\}$ by

$$|\tilde{\xi}_i^1 - \tilde{\xi}_i^0| \leq c\|m^1 - m^0\|_{L^2}.$$

We first show that $\{\tilde{\xi}^k\}_{k\geq 0} \subset L^\infty([0,S];\mathbb{R}^n)$ is a Cauchy sequence. By following the same reasoning as in (4.89), for every $k \geq 1$ we have that

$$\begin{aligned} \|m^k(\cdot,t) - m^{k-1}(\cdot,t)\|_{L^2}^2 &\leq ce^{(2+\beta)S}\int_0^t \|b(m^k)(\cdot,s) - b(m^{k-1})(\cdot,s)\|_{L^2}^2 ds \\ &\leq ce^{(2+\beta)S} 2\frac{1}{1-c_*^2 C\Delta(\epsilon)}\delta(\epsilon). \end{aligned} \tag{4.91}$$

Therefore, since $\|m^k - m^{k-1}\|_{L^2}$ is small, given a mixed contour Γ_i we have that:

$$|\tilde{\xi}_i^k - \tilde{\xi}_i^{k-1}| \leq C\|m^k - m^{k-1}\|_{L^2}. \tag{4.92}$$

For the difference between the two forces $b(m^k)$ and $b(m^{k-1})$, from (4.87) and (4.88) we have:

$$\begin{aligned} \int_0^t \|b(m^k) - b(m^{k-1})\|_{L^2}^2 ds &= \int_0^t\int_{\mathbb{R}}\left(\left(\frac{1-\bar{m}_{\tilde{\xi}^{k-1}}^2}{8}\right)^{\frac{1}{2}} - \left(\frac{1-\bar{m}_{\tilde{\xi}^{k-2}}^2}{8}\right)^{\frac{1}{2}}\right)^2 b_1(x,s)^2 dx\, ds \\ &\leq \frac{1}{8}\int_0^t\int_{\mathbb{R}} |\bar{m}_{\tilde{\xi}^{k-1}}^2 - \bar{m}_{\tilde{\xi}^{k-2}}^2|\, b_1(x,s)^2 dx\, ds \\ &\leq \frac{(\Delta(\epsilon))^2}{4}\sum_{i=1}^n \int_0^t\int_{\Gamma_i} |\bar{m}_{\tilde{\xi}^{k-1}} - \bar{m}_{\tilde{\xi}^{k-2}}| \mathbf{1}_{[|b(\phi)|\leq\Delta(\epsilon)]} dx\, ds \\ &\leq \frac{(\Delta(\epsilon))^2}{2} nS\|\bar{m}'\|_{L^1} \sup_{0\leq s\leq t}\|\tilde{\xi}^{k-1} - \tilde{\xi}^{k-2}\|(s) \end{aligned} \tag{4.93}$$

In the above computations we exploited the fact that m^k and m^{k-1} have the same number of contours and their centers are close to each other due to (4.92). We combine (4.91), (4.92), (4.93) and for ϵ sufficiently small we obtain a contraction:

$$\sup_t \|\tilde{\xi}^k - \tilde{\xi}^{k-1}\| \leq L\sup_t \|\tilde{\xi}^{k-1} - \tilde{\xi}^{k-2}\|$$

where $L = C\|\bar{m}'\|_{L^1} e^{\beta S}\Delta^2 nS < 1$.

Similarly, using the same estimates we can show that the sequences $\{m^k\}_k$ and $\{\phi_1^k\}_k$ are Cauchy in the norm $\sup_t(\|\cdot\|_{W^{1,1}})$ and using a standard argument we can show that the limit point satisfies the system.

4.6 L^1 and L^2 bounds on the centers

We denote

$$\mathcal{N} = \{m \in L^\infty(\mathbb{R},[-1,1]) : \limsup_{x\to-\infty} m(x) < 0;\ \liminf_{x\to+\infty} m(x) > 0\}$$

and define the δ neighborhood of $\mathcal{M}^{(1)} := \{\bar{m}_\xi,\ \xi \in \mathbb{R}\}$ by

$$\mathcal{M}_\delta^{(1)} = \bigsqcup_{\xi \in \mathcal{R}} \{m \in L^\infty(\mathcal{R}, [-1,1]) :\ \|m - \bar{m}_\xi\|_{L^2} < \delta\}.$$

Lemma 4.15. *Any $m \in \mathcal{N}$ has a center. Moreover, there are positive constants c and δ so that any $m \in \mathcal{M}_\delta^{(1)}$ has a unique center $\xi(m)$. Furthermore, for any $n \in \mathcal{M}_\delta^{(1)}$ with $\|m - n\|_{L^1}$ small we have:*

$$|\xi(m) - \xi(n)| \le c\|m - n\|_{L^1}.$$

The same result also holds for the $\|\cdot\|_{L^2}$ norm.

Proof. From the definition of a center it suffices to find a ξ such that

$$(m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} = 0. \tag{4.94}$$

The function $\xi \mapsto (m, \bar{m}'_\xi)_{L^2(d\nu_\xi)}$ is a continuous function and by the definition of $\mathcal{N}$ we have that

$$\limsup_{x \to -\infty} (m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} < 0;\ \liminf_{x \to +\infty} (m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} > 0.$$

Thus (4.94) has a solution.

To show uniqueness, since the function m is in the δ-ball around some $\bar{m}_{\xi_0}$ (without loss of generality we can also assume that $\xi_0 = 0$), we write

$$m = \bar{m} + \psi,\ \|\psi\|_{L^2(d\nu)} < \delta.$$

Then (4.94) gives $(m, \bar{m}'_\xi)_{L^2(d\nu_\xi)} = -(\psi, \bar{m}'_\xi)_{L^2(d\nu_\xi)}$ and since $\|\psi\|_{L^2(d\nu)} < \delta$, we obtain that

$$|(\psi, \bar{m}'_\xi)_{L^2(d\nu_\xi)}| \le \|\psi\|_{L^2(d\nu)} \|\bar{m}'_\xi \frac{1 - \bar{m}^2}{1 - \bar{m}_\xi^2}\|_{L^2(d\nu)} \le \delta \frac{1}{1 - m_\beta^2} \|\bar{m}'\|_{L^2(d\nu)},\ \text{ for any } \xi \in \mathbb{R}. \tag{4.95}$$

Following [60], Theorem 8.5.1.1, we choose $\delta < \frac{\alpha_0}{\|\bar{m}'_\xi\|_{L^2(d\nu_\xi)}}$ which implies that there is no solution to (4.95) when $|\xi| \ge 1$ and $\|m - \bar{m}\|_{L^2(d\nu)} < \delta$.

Given n with $\|m - n\|_{L^1}$ small, we write: $n = m + \chi,\ \text{with } \|\chi\|_{L^1} < \delta'$. We define

$$g(\xi) := (\bar{m}, \bar{m}'_\xi)_{L^2(d\nu_\xi)} + (\psi, \bar{m}'_\xi)_{L^2(d\nu_\xi)} + (\chi, \bar{m}'_\xi)_{L^2(d\nu_\xi)} \tag{4.96}$$

Then $\xi(n)$ is defined by $g(\xi(n)) = 0$. We have:

$$0 = g(\xi(n)) = (\chi, \bar{m}'_{\xi(m)})_{L^2(d\nu_{\xi(m)})} + \int_{\xi(m)}^{\xi(n)} g'(z)dz$$

Since $|\xi(n)| \leq 1$ and $|\xi(m)| \leq 1$ we have that $|z| \leq 1$, thus $g'(z) \geq \alpha_0/2$. Hence,

$$|\xi(n) - \xi(m)| \leq \frac{2}{\alpha_0} |(\chi, \bar{m}'_{\xi(m)})_{L^2(d\nu_{\xi(m)})}| \leq \frac{2}{\alpha_0} \|\chi\|_{L^1} \| \frac{\bar{m}'_{\xi(m)}}{1 - \bar{m}^2_{\xi(m)}} \|_\infty$$

which concludes the proof. Alternatively, we can have the following inequality:

$$|\xi(n) - \xi(m)| \leq \frac{2}{\alpha_0} |(\chi, \bar{m}'_{\xi(m)})_{L^2(d\nu_{\xi(m)})}| \leq \frac{2}{\alpha_0} \|\chi\|_{L^2(d\nu)} \|\bar{m}'_{\xi(m)}\|_{L^2(dx)},$$

which concludes the proof for the case of the L^2 norm as well. $\square$

4.7 Asymptotic analysis of $\mathcal{H}$

For $\mathcal{H}$ given in (3.23) we have that uniformly on $u \in [-1, 1]$ and $w \in (-1, 1)$:

$$\lim_{|b| \to \infty} \frac{\mathcal{H}(b, u, w)}{|b| \log(|b| + 1)} = \frac{1}{2} \qquad \text{and} \qquad \lim_{|b| \to 0} \frac{\mathcal{H}(b, u, w)}{b^2} = \frac{1}{4(1 + uw)}.$$

Moreover, for the choice of $\Delta(\epsilon)$ in (4.36), in the case $|b| \leq \Delta(\epsilon)$, we have that:

$$|\mathcal{H}(b, u, w) - \frac{1}{4(1 + uw))} b^2| \leq C\, |b|^3 \leq C\, \Delta(\epsilon)^3,$$

for some $C > 0$. Thus, for b_1 defined in (4.37), using (4.41) we have that for the same constant $C > 0$ the following hold:

$$\int_{\{|b| \leq \Delta(\epsilon)\}} |\alpha(x,t) b_1(x,t)|^2 dx\, dt \leq \frac{1}{1 - c_*^2 C \Delta(\epsilon)} \int_{\{|b| \leq \Delta(\epsilon)\}} \mathcal{H}(b, u, w) dx\, dt \tag{4.97}$$

and

$$\begin{aligned} \int_{\{|b| \leq \Delta(\epsilon)\}} \left|\mathcal{H}(b, u, w) - \frac{1}{4(1 + uw)} b_1^2\right| dx\, dt &\leq C\Delta(\epsilon) \int_{\{|b| \leq \Delta(\epsilon)\}} b^2(x,t)\, dx\, dt \\ &\leq c_*^2 C \Delta(\epsilon) \int_{\{|b| \leq \Delta(\epsilon)\}} |\alpha(x,t) b(x,t)|^2\, dx\, dt. \end{aligned}$$

Adding and subtracting $\int_{\{|b| \leq \Delta(\epsilon)\}} \mathcal{H}(b, u, w) dx\, dt$, for ϵ small enough it is further implied that

$$\int_{\{|b| \leq \Delta(\epsilon)\}} \left|\mathcal{H}(b, u, w) - \frac{1}{4(1 + uw)} b_1^2\right| dx\, dt \leq \frac{c_*^2 C \Delta(\epsilon)}{1 - c_*^2 C \Delta(\epsilon)} \int_{\{|b| \leq \Delta(\epsilon)\}} \mathcal{H}(b, u, w) dx\, dt, \tag{4.98}$$

which is small as $\epsilon \to 0$ since the cost is bounded by P and $\Delta(\epsilon) \to 0$.

Part II

Simple Symmetric Exclusion Process Driven by Current Reservoirs

Chapter 5

Introduction

One of the most important models on Interacting Particle systems is the *exclusion process*. This is a continuous-time lattice model $\{\eta_t\}_{t\geq 0}$ with at most one particle per site and the simplest imaginabel is considered: whenever a particle tries to jump to a site that is already occupied, that jump is suppressed. Let us give a precise description of the process. For simplicity, the analysis we are after is restricted to $d = 1$. Let us consider a system that evolves in a one-dimensional torus $\mathbb{T}$. We fix a positive integer N, that represents the inverse of the distance between the lattice sites and that will eventually increase to infinity. Then, the microscopic domain has the form $\mathbb{T}_N := \{0, 1, \ldots, N-1\}$. A macroscopic point $r \in \mathbb{T}$ corresponds to a point x at a microscopic scale, if $Nr \in [x, x+1)$, and therefore we could write $x = [r]_N$ (reciprocally, each site x is associated to a macroscopic point x/N in $\mathbb{T}$). The state space is $E = \{0,1\}^{\mathbb{T}_N}$ with particle configurations $\eta = \{\eta(x) : x \in \mathbb{T}_N\}$. The *continuous time stochastic process* $\{H_t\}_{t\geq 0}$ is a family of E-valued random variables $H_t(\eta) = \eta_t$ (canonical projections). The information about the process over time is given by the natural filtration of the probability space, $\{\mathcal{F}_t\}_{t\geq 0}$, which is an increasing sequence of σ-algebras $\mathcal{F}_t$ with

$$\mathcal{F}_t = \sigma\{H_s^{-1}(A) : 0 \leq s \leq t, A \in \mathcal{B}(E)\},$$

where $\mathcal{B}(E)$ is the Borel σ-algebra on E. The time evolution of the system can be described as follows. We consider the probability space $(\mathcal{D}([0,\infty), E), \mathcal{F}, \mathbb{P})$, where $\mathcal{D}([0,\infty), E)$ is the Skorohod space of cadlag trajectories, (continuous from the right and with limits from the left), $\mathcal{F}$ the Borel σ-algebra induced by the Skorohod topology (see [49]) and $\mathbb{P}$ is probability measure on $\mathcal{D}([0,\infty), E)$. We denote by $\{\mathbb{P}^\eta : \eta \in E\}$ the normal Markov process associated to the Feller semigroup $\{S(t)\}_{t\geq 0}$, that is a family of probability measures defined on $\mathcal{D}([0,\infty), E)$ such that

- $\mathbb{P}^{\eta}[\eta_0 = \eta] = 1$ for all $\eta \in E$,
- For all $A \in \mathcal{F}$, $\eta \mapsto \mathbb{P}^{\eta}[A]$ is measurable,
- For all $\eta \in E$, $f \in C(E)$, where $C(E)$ is the space of real-valued, continuous functions on E, and $s, t \geq 0$

$$\mathbb{E}_{\eta}[f(\eta_{t+s})|\mathcal{F}_s] = (S(t)f)(\eta_s), \qquad \mathbb{P}^{\eta} \text{ a.s.}$$

(see [54], Chapter I or [50], Chapter 2). Then, it is easy to check that the operator

$$Lf(\eta) = \sum_{x,y\in\mathbb{T}_N} c(x,y,\eta)\left[f(\eta^{x,y}) - f(\eta)\right], \quad \eta \in E$$

is a Markov pre-generator and its closure is a Markov generator, where f is a cylindrical function (the continuous functions which depend on the configuration η only through a finite number of variables $\eta(x)$), the jump of a particle from x to y with $x, y \in \mathbb{T}_N$ is given by the configuration

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & , z \neq x, y, \\ \eta(x) - 1 & , z = x, \\ \eta(y) + 1 & , z = y \end{cases} \tag{5.1}$$

and

$$c(x,y,\eta) := p(x,y)\eta(x)(1-\eta(y)), \qquad x, y \in \mathbb{T}_N,$$

is the jump rate from x to y. $p(x,y)$ is the *transition rate from x to y*, which are considered *irreducible*: for any pair $x, y \in \mathbb{T}_N$, there exist a number $M \geq 1$ and a sequence $x_0 = x, \ldots, x_M = y$, such that $p(x_i, x_{i+1}) > 0$ for every $1 \leq i \leq M-1$. The Feller semigroup $\{S(t)\}_{t\geq 0}$ on $C(E)$ is associated to the generator L through the Hille-Yosida theorem. When $p(x,y) > 0$ only if x, y are n.n, the process is called *simple*. On top of that, if $p(x,y) = p(y,x)$ the process is also *symmetric*, otherwise it is *assymetric*.

Definition 5.1. A homogeneous Bernoulli product measure on E, and of parameter $\alpha \in [0,1]$, ν_α, is a product and translation invariant measure. Moreover, for a function ρ :

$\mathbb{T}_N \to [0,1]$, ν_ρ is a Bernoulli product on E, if for all $k \in \mathbb{N}$ and $x_1, \dots, x_k \in \mathbb{T}_N$ mutually different $n_1, \dots, n_k \in \{0,1\}$

$$\nu_\rho[\eta(x_1) = n_1, \dots, \eta(x_k) = n_k] = \prod_{i=1}^{k} \nu_\rho^1[\eta(x_i) = n_i],$$

where the *single-site marginals* are given by

$$\nu_\rho^1[\eta(x_i) = 1] = \rho(x_i) \text{ and } \nu_\rho^1[\eta(x_i) = 0] = 1 - \rho(x_i).$$

Therefore, the variables $\{\eta(x) : x \in \mathbb{T}_N\}$ are independent, and especially in the second part of the definition each variable $\eta(x)$ is distributed by Bernoulli with density $\rho(x)$ and with local density $\nu(\eta(x)) = \rho(x)$. A first important result (see [49]) is the following:

Proposition 5.2. *The homogeneous Bernoulli measures $\{\nu_\alpha : \alpha \in [0,1]\}$ on E (see Definition 5.1) are invariant for simple exclusion processes.*

5.0.1 Hydrodynamic Limit

Having gone through the microscopic analysis, we would like to understand the collective behaviour of the time evolution of the systems. In essence, we ask for the macroscopic evolution of the density profile. By a suitable space-time scaling, the study of the limit of the time evolution of the spatial density of particles leads to the *hydrodynamic limit*, which is usually characterised by the (weak) solution of some partial differential equation (PDE), called the *hydrodynamic equation*. We have already seen the space scaling. To rescale a macroscopic time t, we introduce a function $\theta(N)$ that depends on N and the microscopic time is given by $\theta(N)t$. Depending on the model, the function $\theta(N)$ should be different. For example, for the SSEP we need parabolic scaling $\theta(N) = N^2$, while for ASEP, the hyperbolic scaling $\theta(N) = N$ is the suitable one to see the hydrodynamic behaviour.

Examples of hydrodynamic limits

Here we present some results regarding the hydrodynamic limit. We start with the hydrodynamic behaviour of SSEP and ASEP, and then we proceed to the cases where the system is in contact with reservoirs. Proving these results would be omitted, as it goes beyond the scope of the thesis. However, we would like to mention that there is a well-established theory dealing with these kind of problems, such as *Entropy Method, the Relative Entropy Method, Replacement Lemmas, non-gradient techniques and attractiveness techniques* (see for example [49], [50], [61]).

We start by fixing some notation. For a given configuration η, we define the *empirical measure* π^N as the positive measure on $\mathbb{T}$, which gives to each particle a mass $1/N$, namely

$$\pi^N(\eta, dr) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x) \delta_{\frac{x}{N}}(dr)$$

where δ_r is the Dirac measure at r. Then, the time empirical measure is defined as

$$\pi_t^N(\eta, dr) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \delta_{\frac{x}{N}}(dr).$$

Let $\{\mu^N\}_{N\geq 1}$ be a sequence of probability measures on E and $\rho_0 : \mathbb{T} \to [0,1]$ an initial profile.

Definition 5.3. We say that a sequence $\{\mu^N\}_{N\geq 1}$ *is associated to* ρ_0, if for every function $H : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$,

$$\lim_{N\to\infty} \mu^N \left[\eta : \left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(\frac{x}{N})\eta(x) - \int_{\mathbb{T}} H(r)\rho_0(r)dr \right| > \delta \right] = 0 \tag{5.2}$$

This ensures that the empirical measure at time 0 satisfies a law of large numbers. The addressed question is the following: if at time 0, $\{\mu^N\}_{N\geq 1}$ is associated to some initial profile ρ_0, does $\pi^N_{\theta(N)t}(\eta, dr)$ converge in an appropriate way to a profile ρ_t at time t? We answer the question through the following examples.

Example 1. The theorem gives the hydrodynamic limit of the Symmetric simple exclusion process.

Theorem 5.4. *Consider the symmetric simple exclusion process on $\mathbb{T}_N^d$ where $d \geq 1$. Then, starting from the local equilibrium measure $\{\mu^N\}_{N\geq 1}$ associated with profile ρ_0, the empirical measure $\pi^N_{N^2 t}(\eta, dr)$ converges weakly in probability (as in* (5.2)*) to the measure $\rho(r,t)dr$, where $\rho(r,t)$ satisfies the hydrodynamic equation:*

$$\begin{cases} \partial_t \rho(r,t) = \frac{1}{2}\Delta\rho(r,t), & \textit{for } t > 0,\ r \in (0,1), \\ \rho(r,0) = \rho_0(r), & \textit{for } r \in [0,1]. \end{cases}$$

For the proof, see for example [49], Chapter 4.

Example 2. Another example that the hydrodynamic limit has been studied is for ASEP (see [61]). We recall that from a microscopic point of view, a simple exclusion process takes place in the bulk, where a particle waits exponential time with parameter one and attempts a transition of one unit to the right with probability p if that site is vacant, otherwise it remains where it was. The same mechanism happens to the left, but with probability q. The probabilities p, q satisfy $p+q = 1$ and $p \neq \frac{1}{2}$ and that is why the model is called asymmetric. Having the assumption of starting from the local equilibrium measure $\{\mu^N\}_{N\geq 1}$, associated with profile ρ_0 plus some extra hypotheses, then the empirical measure under hyperbolic scaling $\theta(N) = N$ instead of parabolic converges weakly in probability to a measure $\rho(r,t)dr$, where $\rho(r,t)$ is the entropy solution of the hyperbolic equation, known as the *inviscid Burgers equation with boundary conditions*:

$$\begin{cases} \partial_t \rho(r,t) + (p-q)(1-2\rho(r,t))\nabla\rho(r,t) = 0, & \text{for } t > 0,\ r \in (0,1), \\ \rho(r,0) = \rho_0(r), & \text{for } r \in [0,1]. \end{cases}$$

Example 3. The case of boundary which is driven by the simple symmetric exclusion process is also interesting. This particle system is a simple model of mass transfer between reservoirs of different densities. In fact, the system is in contact with reservoirs, imposing a gradient on the conserved quantities of the system. In [55], the authors study the simple exclusion process in $\Lambda_N := \{1, \dots, N-1\}$ and at the left boundary, particles are created with rate α and annihilated with rate $1-\alpha$, while on the right boundary this is done with rates β and $1-\beta$ with $0 \leq \alpha \leq \beta \leq 1$. Formally, this is a Markov process on $\{0,1\}^{\Lambda_N}$, whose generator is given by

$$\begin{aligned} L_N f(\eta) \quad := \quad & \frac{1}{2}\sum_{x=1}^{N-2}[f(\eta^{x,x+1}) - f(\eta)] + \\ & + \{(1-\alpha)\eta(1) + \alpha(1-\eta(1))\}\,[f(\eta^1) - f(\eta)] + \\ & + \{(1-\beta)\eta(N-1) + \beta(1-\eta(N-1))\}\,[f(\eta^{N-1}) - f(\eta)] \end{aligned}$$

where the configuration η^x corresponds to annihilation/creation of a particle at the site x,

$$\eta^x(z) = \begin{cases} \eta(z) & , z \neq x, \\ 1-\eta(x) & , z = x. \end{cases} \tag{5.3}$$

This finite state Markov process is irreducible and therefore has a unique stationary measure, denoted by $\nu^N_{\alpha,\beta}$. If $\alpha = \beta$, then an elementary computation shows that ν^N_α is the

Bernoulli product measure on $\{0,1\}^{\Lambda_N}$ with density α. From Proposition 5.2, ν_α^N is invariant measure, and therefore the process is reversible with respect to this stationary state. On the other hand, if $\alpha < \beta$, it is known since [64] that the invariant state has long range correlations. However, in any case we can compute linear profiles that are associated to invariant measures, namely the linear profile

$$\bar{\rho}(r) = (\beta - \alpha)r + \alpha$$

is obtained by computing $\mathbb{E}_{\nu^N_{\alpha,\beta}}(\eta([r]_N))$ in the limit. Also note that $\bar{\rho}(r)$ is the stationary solution of the hydrodynamic equation given in Theroem 5.5 below. Denote by $\mathbb{P}_{\mu^N}$, the probability on the path space $\mathcal{D}(\mathbb{R}_+,\{0,1\}^{\Lambda_N})$ induced by the Markov process with generator L_N and the initial measure μ^N, then the hydrodynamic behaviour for this model is summarised by the following theorem (see [49] and [56]):

Theorem 5.5. *Starting from the local equilibrium measure $\{\mu^N\}_{N\geq 1}$, associated with a profile ρ_0, then*

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}\left[\eta. : \left| \frac{1}{N}\sum_{x\in\Lambda_N} H(\frac{x}{N})\eta_{N^2t}(x) - \int_{\mathbb{T}} H(r)\rho(r,t)dr\right| > \delta\right] = 0$$

where $\rho(r,t)$ satisfies the hydrodynamic equation:

$$\begin{cases} \partial_t\rho(r,t) = \Delta\rho(r,t), & \textit{for } t>0,\ r\in(0,1), \\ \rho(r,0) = \rho_0(r), & \textit{for } r\in[0,1], \\ \rho(0,t) = \alpha, \quad \rho(1,t)=\beta, & \textit{for } t>0. \end{cases}$$

Example 4. One last example that we would like to stress is the hydrodynamic limit of the simple exclusion process with *slow boundary*. In [2], the authors consider the SSEP in Λ_N with slow boundaries, that is particles are created or annihilated at the boundary sites 1 and $N-1$ at a rate proportional to $N^{-\theta}$, where $\theta \geq 0$. To be precise, particles can enter the system at site 1 with rate $\frac{c\alpha}{N^\theta}$, or leave with rate $\frac{c(1-\alpha)}{N^\theta}$, while at the site $N-1$ the behaviour is similar, but the rates are $\frac{c\beta}{N^\theta}$ for entering the system and $\frac{c(1-\beta)}{N^\theta}$ for leaving the system. $\alpha,\beta\in(0,1)$, $c>0$ and $\theta\geq 0$. This is a Markov process on $\{0,1\}^{\Lambda_N}$ whose generator is given by

$$\begin{aligned} L_N f(\eta) \ &:= \ \sum_{x=1}^{N-2}[f(\eta^{x,x+1}) - f(\eta)] + \\ &+cN^{-\theta}\left\{(1-\alpha)\eta(1) + \alpha(1-\eta(1))\right\}[f(\eta^1) - f(\eta)] + \\ &+cN^{-\theta}\left\{(1-\beta)\eta(N-1) + \beta(1-\eta(N-1))\right\}[f(\eta^{N-1}) - f(\eta)] \end{aligned}$$

To state the hydrodynamic behaviour of this model, we need to establish some notation. We fix a time $T > 0$ and we denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(0,1)$, $C_k^\infty(0,1)$ is the set of all smooth functions defined in $(0,1)$ with compact support, $\mathcal{H}^1(0,1)$ is the set of all locally integrable functions $g : (0,1) \to \mathbb{R}$, such that there exists a function $\partial_r g \in L^2(0,1)$ satisfying $\langle \phi, \partial_r g \rangle = -\langle \partial_r \phi, g \rangle$, for all $\phi \in C_k^\infty(0,1)$, $L^2(0,T;\mathcal{H}^1(0,1))$ is the set of measurable functions $f : [0,T] \to \mathcal{H}^1(0,1)$ such that $\int_0^T \|f\|^2_{\mathcal{H}^1(0,1)} dt < \infty$, where $\|f\|_{\mathcal{H}^1(0,1)} = \left(\|f\|_{L^2(0,1)} + \|\partial_r f\|_{L^2(0,1)} \right)^{1/2}$, $C_0^{1,2}([0,T] \times [0,1])$ is the set of all functions H such that $H(0,t) = H(1,t) = 0$ for all $t \in [0,T]$ and finally, $C^{n,m}([0,T] \times [0,1])$ is the set of all functions that defined on $[0,T] \times [0,1]$, that are n times differentiable w.r.t the first variable and m times differentiable w.r.t the second variable. In [2], the authors prove the following result:

Theorem 5.6. *Starting from the local equilibrium measure $\{\mu^N\}_{N \geq 1}$, associated with a profile $\rho_0 : [0,1] \to [0,1]$, for each $t \in [0,T]$, every $\delta > 0$ and for all functions $H \in C([0,1])$, we have that*

$$\lim_{N \to \infty} \mathbb{P}_{\mu^N} \left[\eta_\cdot : \left| \frac{1}{N} \sum_{x \in \Lambda_N} H(\frac{x}{N}) \eta_{N^2 t}(x) - \int_{\mathbb{T}} H(r) \rho(r,t) dr \right| > \delta \right] = 0$$

where

1. *If $\theta = 1$, then $\rho(r,t)$ is a weak solution of the heat equation with Robin boundary conditions, that is*

$$\begin{cases} \partial_t \rho(r,t) = \Delta \rho(r,t), & \textit{for } t > 0,\ r \in (0,1), \\ \partial_r \rho(0,t) = c(\rho(0,t) - \alpha), & \textit{for } t \geq 0, \\ \partial_r \rho(1,t) = c(\beta - \rho(1,t)), & \textit{for } t \geq 0, \\ \rho(r,0) = \rho_0(r), & \textit{for } r \in [0,1]. \end{cases}$$

   Equivalently, $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$ and ρ satisfies

$$\begin{aligned} \langle \rho(\cdot,t), H(\cdot,t) \rangle - \langle \rho_0, H_0 \rangle = & \int_0^t \langle \rho(\cdot,s), (\partial_s + \Delta) H(\cdot,s) \rangle ds \\ & + \int_0^t (\rho(0,s) \partial_r H(0,s) - \rho(1,s) \partial_r H(1,s)) ds \\ & + c \int_0^t (H(0,s)(\alpha - \rho(0,s)) - H(1,s)(\beta - \rho(1,s))) ds \end{aligned}$$

   for all $t \in [0,T]$ and $H \in C^{1,2}([0,T] \times [0,1])$.

2. *If $\theta \in [0,1)$, then $\rho(r,t)$ is a weak solution of the heat equation with Dirichlet boundary conditions, that is*

$$\begin{cases} \partial_t \rho(r,t) = \Delta\rho(r,t), & \text{for } t>0,\ r\in(0,1), \\ \rho(0,t)=\alpha, \quad \rho(1,t)=\beta, & \text{for } t\geq 0, \\ \rho(r,0)=\rho_0(r), & \text{for } r\in[0,1]. \end{cases}$$

Equivalently, $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$ and ρ satisfies

$$\begin{aligned} \langle \rho(\cdot,t), H(\cdot,t)\rangle - \langle \rho_0, H_0\rangle &= \int_0^t \langle \rho(\cdot,s), (\partial_s+\Delta)H(\cdot,s)\rangle ds \\ &\quad - \int_0^t (\beta H(1,s) - \alpha H(0,s))ds, \end{aligned}$$

for all $t\in[0,T]$ and $H\in C_0^{1,2}([0,T]\times[0,1])$.

3. *If $\theta > 1$, then $\rho(r,t)$ is a weak solution of the heat equation with Neumann boundary conditions, that is*

$$\begin{cases} \partial_t \rho(r,t) = \Delta\rho(r,t), & \text{for } t>0,\ r\in(0,1), \\ \partial_r\rho(0,t)=0, \quad \partial_r\rho(1,t)=0, & \text{for } t\geq 0, \\ \rho(r,0)=\rho_0(r), & \text{for } r\in[0,1]. \end{cases}$$

Equivalently, $\rho \in L^2(0,T;\mathcal{H}^1(0,1))$ and ρ satisfies

$$\begin{aligned} \langle \rho(\cdot,t), H(\cdot,t)\rangle - \langle \rho_0, H_0\rangle = &\int_0^t \langle \rho(\cdot,s), (\partial_s+\Delta)H(\cdot,s)\rangle ds \\ &- \int_0^t (\rho(1,s)\partial_r H(1,s) - \rho(0,s)\partial_r H(0,s))ds, \end{aligned}$$

for all $t\in[0,T]$ and $H\in C^{1,2}([0,T]\times[0,1])$.

5.0.2 Fluctuations

The states of real systems exhibit small deviations and incongruous oscillations around the pattern predicted by the hydrodynamic limit. While the hydrodynamic limit is a Law of Large Numbers (L.L.N.) type-theorem, the scaling limit for how the discrete system oscillates around its hydrodynamic limit is usually referred to as *fluctuations*, and is a Central Limit Theorem (C.L.T.). The analysis of fluctuations is interesting, not only at the non-equilibrium case, but also at equilibrium. However, equilibrium fluctuations of particle systems are relatively well understood, (see [49]) while non-equilibrium fluctuations have only been derived for very few models, due to the long range space-time correlations.

Analysis of fluctuations

For the present analysis in order to discuss a general strategy of proving fluctuations, we consider the path space $\mathcal{D}([0,\infty), \mathcal{J}'(\mathbb{R}))$ containing distribution valued trajectories on the Schwarz space $\mathcal{J}(\mathbb{R})$, that is the space of the rapidly decreasing test functions.

Definition 5.7. We define the density fluctuation field $Y_{\cdot}^N(H)$ as the time-trajectory of linear functionals acting on functions $H \in \mathcal{J}$ as

$$Y_t^N(H) := \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} H(\epsilon x) \left[\eta_t(x) - \mathbb{E}_{\mu^N}^N(\eta_t(x))\right], \text{ for all } t \geq 0, \tag{5.4}$$

where $\mathbb{E}_{\mu^N}^N$ is the expectation which corresponds to the probability $\mathbb{P}_{\mu^N}$, defined on the path space $\mathcal{D}(\mathbb{R}_+, \{0,1\}^{\Lambda_N})$, induced by the Markov process with generator L_N and the initial measure μ^N. We denote by $\mathbb{Q}^N$ the probability measure on $\mathcal{D}([0,\infty), \mathcal{J}'(\mathbb{R}))$, induced by the density fluctuation field $Y_{\cdot}^N$ and μ^N.

Remark 5.8. In the equilibrium case, the fluctuation field is given by the choice $\mu^N = \nu_\rho^N$, where ν_ρ^N is the Bernoulli product measure with parameter $\rho \in (0,1)$, and therefore $\mathbb{E}_{\nu_\rho^N}^N(\eta_{N^2 t}(x)) = \rho$ for all $x \in \Lambda_N$ and for all $t \geq 0$. The probability measure on $\mathcal{D}([0,\infty), \mathcal{J}'(\mathbb{R}))$ induced by the density fluctuation field $Y_{\cdot}^N$ and ν_ρ^N is denoted by $\mathbb{Q}_\rho^N$.

Having established a general setting around the equilibrium fluctuation field, two questions arise:

1. Does the process $\{\mathcal{D}([0,\infty), \mathcal{J}'(\mathbb{R})), \mathbb{Q}^N\}$ converges weakly as $N \to \infty$ to a process $\{\mathcal{D}([0,\infty), \mathcal{J}'(\mathbb{R})), \mathbb{Q}\}$?
2. If we have such a limiting process, could we characterise it?

The first question is equivalent to proving tightness of the measures $\mathbb{Q}^N$, while a general approach which employs Holley and Stroock's results in [45] become standard in characterising the limits.

Tightness

We fix a time $T > 0$, and let $\{\mathbb{Q}^N_{\mu^N}\}_{N\geq 1}$ be measures on $\mathcal{D}([0,T],\mathcal{S}')$ induced by a sequence $\{X_t^N : t \in [0,T]\}_{N\geq 1}$ of $\mathcal{S}'$-valued processes and arbitrary initials measures $\{\mu^N\}_{N\geq 1}$. The space $\mathcal{S}'$ is the topological dual space of an arbitrary space $\mathcal{S}$. In general, there are several ways to prove tightness. We refer to three alike tightness-type results, which have been mostly used in the literature, and each is stated for a different space $\mathcal{S}$.

- $\mathcal{S} = \mathcal{H}_k$ and $\mathcal{S}' = \mathcal{H}_{-k}$ for a positive number k as defined below (see [49], Chapter 11).

  For any integer $z \geq 0$, we define $h_z : \mathbb{T} \to \mathbb{R}$ by

$$h_z(r) = \begin{cases} \sqrt{2}\cos(2\pi z r), & if\ z > 0, \\ 1, & if\ z = 0, \\ \sqrt{2}\sin(2\pi z r), & if\ z < 0 \end{cases}$$

  Every function $f \in L^2(\mathbb{T})$ can be written as

$$f = \sum_{z\in\mathbb{Z}} \langle f, h_z\rangle_{L^2(\mathbb{T})} h_z$$

  as by Sturm-Liouville theory, it is known that $\{h_z : z \in \mathbb{Z}\}$ is an orthogonal basis of $L^2(\mathbb{T})$. By $\langle \cdot,\cdot\rangle$ we denote the inner product of $L^2(\mathbb{T})$. Let us consider now the positive, symmetric linear operator

$$\mathcal{L} = 1 - \Delta$$

  One can show that h_z are its eigenvectors and $\lambda_z = 1 + 4\pi^2\|z\|^2$ are its eigenvalues. For a positive integer k, $\mathcal{H}_k$ is the Hilbert space obtained by the completion of $C^\infty(\mathbb{T})$ equipped with the inner product

$$\langle f, g\rangle_{\mathcal{H}_k} = \langle f, \mathcal{L}^k g\rangle_{L^2(\mathbb{T})}$$

  and in particular,

$$\langle f, g\rangle_{\mathcal{H}_k} = \sum_{z\in\mathbb{Z}} \langle f, h_z\rangle_{L^2(\mathbb{T})} \langle g, h_z\rangle_{L^2(\mathbb{T})}\, \lambda_z^k$$

  and therefore

$$\mathcal{H}_k = \{f \in L^2(\mathbb{T}) : \sum_{z\in\mathbb{Z}} \langle f, h_z\rangle^2_{L^2(\mathbb{T})}\, \lambda_z^k < \infty\}$$

Furthermore,

$$L^2(\mathbb{T}) \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots$$

The topological dual of $\mathcal{H}_k$ relatively to $\langle \cdot, \cdot \rangle$ is denoted by $\mathcal{H}_{-k}$, and it can be obtained by the completion of $L^2(\mathbb{T})$ w.r.t the inner product obtained from the quadratic form

$$\langle f, f \rangle_{\mathcal{H}_{-k}} = \sup_{g \in \mathcal{H}_k} \left\{ 2\langle f, g \rangle_{L^2(\mathbb{T})} - \langle g, g \rangle_{\mathcal{H}_k} \right\}$$

Similarly,

$$\langle f, g \rangle_{\mathcal{H}_{-k}} = \sum_{z \in \mathbb{Z}} (f, h_z)(g, h_z) \lambda_z^{-k}$$

where by $(\cdot, \cdot)$ we denote the duality pairing between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$ and therefore

$$\mathcal{H}_{-k} = \left\{ \{(f, h_z)\}_{z \in \mathbb{Z}} : \sum_{z \in \mathbb{Z}} (f, h_z)\, \lambda_z^{-k} < \infty \right\}$$

Furthermore,

$$\cdots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset L^2(\mathbb{T}) \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \cdots$$

The next theorem guarantees tightness of $\{\mathbb{Q}^N_{\mu^N}\}_{N \geq 1}$ on $\mathcal{D}([0, T], \mathcal{H}_{k_0})$, for large enough k_0 endowed with the uniform weak topology: we say that $\{X^N_\cdot\}_{N \geq 1}$ converges to a path $X_\cdot$, for all $H \in \mathcal{H}_{k_0}$

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} \left| X^N_t(H) - X_t(H) \right| = 0$$

Theorem 5.9. *A sequence $\{\mathbb{Q}^N_{\mu^N}\}_{N \geq 1}$ on $\mathcal{D}([0, T], \mathcal{H}_{k_0})$ is tight if for every $0 \leq t \leq T$, the following hold:*

(i)

$$\lim_{A \to +\infty} \limsup_{N \to \infty} \mathbb{Q}^N_{\mu^N} \left(\sup_{0 \leq t \leq T} \|X^N_t\|_{\mathcal{H}_{-k_0}} > A \right) = 0,$$

(ii) *for every $\delta > 0$,*

$$\lim_{\zeta \to 0} \limsup_{N \to \infty} \mathbb{Q}^N_{\mu^N} \left(\sup_{\substack{|s-t| \leq \zeta, \\ 0 \leq s, t \leq T}} \|X^N_t - X^N_s\|_{\mathcal{H}_{-k_0}} > \delta \right) = 0.$$

For the proof of the theorem we refer the reader to [49], Chapter 11.

- $\mathcal{S} = \mathcal{J}(\mathbb{R})$, the Schwartz space of C^∞, which decay at infinity, together with their derivatives faster than any inverse power. In this case, $\{\mathbb{Q}^N_{\mu^N}\}_{N\geq 1}$ are a family of measures on $\mathcal{D}([0,T], \mathcal{J}'(\mathbb{R}))$ (see for example [21]).

Theorem 5.10. *For every $H \in \mathcal{J}(\mathbb{R})$, the random variables X^N_t, whose distribution is given by $\mathbb{Q}^N_{\mu^N}$, are tight if the following two conditions hold:*

(i)

$$\sup_N \mathbb{Q}^N_{\mu^N}\left(\sup_{0\leq t\leq T} X^N_t(H)^2\right) < \infty,$$

(ii) *for every $\delta > 0$, there are $d > 0$ and N_0 such that for any $N \geq N_0$*

$$\sup_N \mathbb{Q}^N_{\mu^N}\left(\sup_{\substack{|T_1-T_2|\leq d,\\ 0\leq T_1,T_2\leq T}} |X^N_{T_1}(H) - X^N_{T_2}(H)| > \delta\right) < \delta.$$

- $\mathcal{S}$ is any Frechét space, that is a Hausdorff space where its topology may be induced by a countable family of semi-norms. The Mitoma's criterium given below is quite powerful, in the sense that if the space is a Frechét space, then it is enough to prove tightness on $\mathcal{D}([0,\infty), \mathbb{R})$.

Theorem 5.11. *[1, Mitoma's Criterium] If $\mathcal{S}$ is a Frechét space, then the sequence $\{X_t : t \geq 0\}_{N\geq 1}$ whose distribution is given by $\mathbb{Q}^N_{\mu^N}$,is tight w.r.t Skorohod topology of $\mathcal{D}([0,\infty), \mathcal{S}')$, if and only if, the sequence $\{X_t(H) : t \geq 0\}_{N\geq 1}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathcal{D}([0,\infty), \mathbb{R})$, for any $H \in \mathcal{S}$.*

Theorem 5.12. *[57, Aldous' Criterium] Let $\{\mathbb{Q}^N_{\mu^N}\}_{N\geq 1}$ be a family of processes on $\mathcal{D}([0,\infty), \mathcal{S}')$. Then, the sequence $\{X_t : t \geq 0\}_{N\geq 1}$, whose distribution is given by $\mathbb{Q}^N_{\mu^N}$,is tight w.r.t Skorohod topology of $\mathcal{D}([0,\infty), \mathbb{R})$, if the following two conditions hold:*

(i)

$$\lim_{A\to+\infty} \limsup_{N\to\infty} \mathbb{Q}^N_{\mu^N}\left(\sup_{0\leq t\leq \tau} X^N_t > A\right) = 0,$$

(ii) *for every $\delta > 0$,*

$$\lim_{\zeta\to 0} \limsup_{N\to\infty} \sup_{\lambda\leq\zeta} \sup_{\tau\in\mathcal{T}_\tau} \mathbb{Q}^N_{\mu^N}\left(|X^N_\tau - X^N_{\tau+\lambda}| > \delta\right) = 0$$

where $\mathcal{T}_T$ is the set of stopping times bounded by T.

Characterisation of the limiting process

As mentioned, in [45] Holley and Stroock provide insight to the characterisation of a limiting process ensured by tightness. Let us fix some ideas. Let $\mathscr{A} : \mathcal{J}(\mathbb{R}) \to \mathcal{J}(\mathbb{R})$ be a bounded linear operator, which admits a non-positive definite self-adjoint extention $\bar{\mathscr{A}}$ on $L^2(\mathbb{R})$. We further assume that there is a strongly continuous semi-group $\{T_t\}_{t\geq 0}$ of bounded linear operators on $\mathcal{J}(\mathbb{R})$ into itself, such that

$$N(T_tH) - N(H) = \int_0^t N(\mathscr{A}T_sH)ds, \qquad t \geq 0$$

for all $N \in \mathcal{J}'(\mathbb{R})$ and $H \in \mathcal{J}(\mathbb{R})$. It is easy to show that $T_tH = e^{t\bar{\mathscr{A}}}H$ almost surely, where $e^{t\bar{\mathscr{A}}}$ denotes the semi-group of self-adjoint contractions on $L^2(\mathbb{R})$ generated by $\bar{\mathscr{A}}$, and $\mathscr{A}T_tH = T_t\mathscr{A}H$. Furthermore, $\mathcal{F}_t = \sigma\left(\{N_s(H) : 0 \leq s \leq t, H \in \mathcal{J}(\mathbb{R})\}\right)$ and $\mathcal{F} = \sigma(\bigcup_{t\geq 0}\mathcal{F}_t)$. Finally, let $\mathscr{B} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a bounded linear operator. We focus on the crucial result of the theory, which is intimately connected to the following:

Theorem 5.13. *[45, Generalised Ornstein-Uhlenbeck Process] Let $\mathbb{Q}$ be a probability measure on $(\mathcal{D}([0,\infty),\mathbb{R}), \mathcal{F})$ such that*

$$F(N_{\tau\wedge t}(H)) - \int_0^{\tau\wedge t} N_s(\mathscr{A}H)F'(N_s(H))ds - \frac{\|\mathscr{B}H\|^2}{2}\int_0^{\tau\wedge t} F''(N_s(H))ds, \tag{5.5}$$

where $N_s(H) = N(T_sH)$, is a martingale w.r.t $(\mathcal{D}([0,\infty),\mathbb{R}), \mathcal{F}_t, \mathbb{Q})$ for all $F \in C_0^\infty(\mathbb{R})$, $H \in \mathcal{J}(\mathbb{R})$ and stopping times τ such that $\sup_\omega \sup_{t\geq 0} |N_{t\wedge\tau(\omega)}(\mathscr{A}H, \omega)| < \infty$. Then for all $F \in C_0^\infty(\mathbb{R})$ and $H \in \mathcal{J}(\mathbb{R})$,

$$F\left(N_t(H) - \int_0^t N_s(\mathscr{A}H)ds\right) - \frac{\|\mathscr{B}H\|^2}{2}\int_0^t F''\left(N_s(H) - \int_0^t N_u(\mathscr{A}H)du\right)ds$$

is a martingale w.r.t $(\mathcal{D}([0,\infty),\mathbb{R}), \mathcal{F}_t, \mathbb{Q})$. Moreover, for all $0 \leq s < t$, $H \in \mathcal{J}(\mathbb{R})$ and $\Gamma \in \mathcal{B}(\mathbb{R})$

$$\mathbb{Q}(N_t(H) \in \Gamma|\mathcal{F}_u) = \int_\Gamma g\left(\int_0^{t-u} \|\mathscr{B}T_sH\|^2 ds, y - N_u(T_{t-u}H)\right)dy, \qquad \mathbb{Q} - \text{a.s.}$$

where

$$g(t,y) = \frac{1}{(2\pi t)^{1/2}}e^{-\frac{y^2}{2t}}.$$

In particular, the condition (5.5) on $\mathbb{Q}$, together with a knowledge of $\mathbb{Q}|\mathcal{F}_0$, uniquely determines $\mathbb{Q}$ on $(\mathcal{D}([0,\infty),\mathbb{R}), \mathcal{F})$. Finally, $\mathbb{Q}$ satisfies this condition, if and only if, the distribution of $N_t - T_tN_0$ also satisfies it.

Remark 5.14. The fact that (5.5) is a martingale w.r.t $(\mathcal{D}([0,\infty),\mathbb{R}),\mathcal{F}_t,\mathbb{Q})$ may allow us to interpret it, by saying that $N_t(H)$ formally satisfies the following stochastic differential equation

$$dN_t(H) = N_t(\mathscr{A}T_tH)dt + d\mathcal{W}_t(\mathscr{B}H) \tag{5.6}$$

where $\mathcal{W}_t$ is a space-time white noise of unit variance. The solution of (5.6) is called *Generalised Ornstein-Uhlenbeck process with characteristics $\mathscr{A}$ and $\mathscr{B}$.*

In general, there is a common strategy in order to characterise the limiting process $\{\mathcal{D}([0,\infty),\mathcal{J}'(\mathbb{R})),\mathbb{Q}\}$ as the Generalised Ornstein-Uhlenbeck process (see for example [33], [25], [34], [62]). We analyse it in detail below. To exploit Theorem 5.13, one has to show that the condition (5.5) holds. To do that, we use the following: for every $F \in C_0^\infty(\mathbb{R})$ and $H \in \mathcal{J}(\mathbb{R})$

$$F(Y^N_{\theta(N)t}(H)) - \int_0^{\theta(N)t} L_N F(Y^N_s(H))ds \tag{5.7}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma(\{\eta_s : s \le t\})$, where $\theta(N)t$ is the microscopic time as we discussed in Sect. 5.0.1, $Y^N_t(H)$ is given by (5.4) and L_N is the generator of a Markov process $\{\eta_t\}_{t\ge 0}$. We compute the quantity inside the integral, and we get

$$L_N F(Y^N_s(H)) = F'(Y^N_s(H))\Lambda^N_s(H) + \frac{1}{2}F''(Y^N_s(H))\Gamma^N_s(H) + K^N_t(H) \tag{5.8}$$

where

$$\Lambda^N_s(H) := (\partial_s + L_N)Y^N_s(H) \tag{5.9}$$

$$\Gamma^N_s(H) := L_N(Y^N_s(H))^2 - 2Y^N_s(H)L_NY^N_s(H). \tag{5.10}$$

By comparing the integral of (5.8) multiplied by $\theta(N)$ and (5.5) with $N_t = Y_t$, where Y_t is the limiting fluctuation field and its existence is ensured by tightness, it is enough to prove that the first and the second term of the r.h.s. of (5.5) converge to the second and the third integral terms respectively, while one has to show that $\theta(N)K^N_t(H)$ is uniformly bounded by $\frac{1}{N}$. In general, to prove convergence is not straightforward. The main difficulty is to "close" the martingale problem, which means that (5.8) is expressed in a closed form w.r.t. Y^N_t. This mainly requires a *Boltzmann-Gibbs principle* type result (see [49], [62], [25]). Moreover, when the system is in contact with reservoirs at the boundaries, one may *choose a suitable space of test functions*. In the explicit computation of $\Lambda^N_s(H)$, there are terms that are already closed with respect to the fluctuation field, while some other terms coming from the boundary, blow up in the limit. By a special choice of test functions we can make these terms vanish in the limit, and therefore we express (5.8) in a closed form (see [33], [34]).

Once the convergence of (5.7) to (5.5) is proven, then by Theorem 5.13, the limiting process is the Ornstein-Uhlenbeck Process. We also note that by imposing some extra assumption about the fluctuation field at time 0 (e.g $\{Y_0^N\}_{N\geq 1}$ converges to a mean-zero Gaussian field Y with a particular covariance, see e.g. [55], [33], [34]), by the third part of Theorem 5.13 we get uniqueness of the limiting process. We refer to [33],[55], [34], [25], [62], [35], [46] and [61] and reference therein for equilibrium and non-equilibrium fluctuations.

In the following, we investigate the non-equilibrium fluctuations of the model described in Sect. 6.1. Our *conjecture* is that the non-equilibrium fluctuations around its hydrodynamic limit (see Sect. 6.2 and [22]) is given by the Ornstein-Uhlenbeck Process with particular characteristics. In Sect. 6.3.2 and 6.3.4 we give detailed computations for the *discrete variance kernel* and the *continuous variance kernel*, while in Sect. 6.4 we compute Λ_t^N and Γ_t^N defined (6.51) for this model. In particular, as we will see in Sect. 6.4, the quadratic variation Γ_t^N converges to a quantity which, after choosing a suitable space of the function $\mathcal{C}$ (see Definition 6.12) and a semigroup T_t, should be the variance of the limiting fluctuation field. We compare that with the solution of the continuous variance kernel equation and this might indicate the extra constraints for the test functions. This is done in Sect. 6.4.1. By computing explicitly Λ_t^N, we have terms that are closed with respect to the fluctuation field Y_t^N and some others that are problematic. In fact, the presence of boundary terms, which are non-linear in this model, does not allow for a direct closure of the above martingales. These terms could be split into two categories. The first is for those that blow up in the limit and the second one is for those that "naturally" vanish thanks to correlation estimates. To deal with the terms of the first category, we use the choice of the space of test functions in conjunction with the choice of the semigroup that the test functions evolve, of Sect 6.4.1. The terms of the second category vanish as $N \to \infty$, once we have appropriately good correlation estimates. However, the already existing n-body correlation estimates or the so-called v-estimates (see [22] and [23]) are not good enough to make them vanish either for the "closure" of the martingale problem or for proving tightness of the process. Thus, an improvement of those estimates is necessary as in [25]. A correlation estimate in space and time may also be needed.

Chapter 6

Non-equilibrium fluctuations for the SSEP driven by current reservoirs

Addendum

Throughout the correction period, we have made a few advances regarding the conjecture, which is stated in Theorem 6.9, as well as to its consequences to the strategy that we discuss in this chapter. More specifically, such a result seemed necessary in order to make vanish particular terms arising from the martingale decomposition (Sect. 6.4, see also remarks at the end of this chapter). We elucidated that the estimate given in the theorem is not correct. Nevertheless, by following similar techniques as in paper [23], we could get sufficient estimates for eliminating each problematic term of the martingale decomposition (see (6.58) and (6.64)) which will be thoroughly discussed in Sect. 6.4. Therefore, apart from this result the rest of the current chapter remains as it is.

6.1 The model

In our model, we consider the one dimensional SSEP in the interval $[-N, N]$, N is a positive integer where (independently) each particle tries to jump to one of the nearest neighbour sites at rate $N^2/2$. In addition, we consider currents that are obtained by driving forces that act on the boundaries. The driving forces could be physically interpreted in terms of reservoirs. In the model, we implement the so called *current reservoirs*, namely we fix the current so that we send in particles from the right, at a rate which according to Fick's law, has to be inversely proportional to the size of the system, and take out particles from the left at the same rate. For this reason, the rate is equal to $Nj/2$, where $j > 0$ is the external parameter which rules the birth-death mechanisms in the right-left boundaries.

There is a significant work in the literature studying problems of mass transport in a diffusive system under the action of the so called *density reservoirs* that is, systems in an interval, where the reservoirs add and subtract particles from the right and left respectively at a unit rate, but instead of keeping fixed the current, they fix given densities close to the right and left. This also leads to a production of current from the high to low densities according to Fick's law.

In [22, 23, 24], the authors study the hydrodynamic limit of the evolution of the density field, they establish the propagation of chaos and they study the stationary density field in the limit of the model. The purpose of [13] is to investigate the behaviour of the non-equilibrium fluctuations for the model. In particular, we wish to show that the sequence of the density fluctuation field is tight, and every limiting point at time t can be written as the sum of a Gaussian random variable and the initial condition. Furthermore, if we assume that the sequence at time 0 converges to a mean-zero Gaussian process, then the limit of the sequence is unique and is given by the *Ornstein-Uhlenbeck Process* with certain boundary conditions.

Let Λ_N be the interval in $\mathbb{Z}$ with endpoints $\pm N$, denoted by $\Lambda_N = [-N, N]$. We fix an integer $K \geq 1$, write $I_+ := [N-K+1, N]$ and $I_- := [-N, -N+K-1]$. Particle configurations are elements η of $\{0,1\}^{\Lambda_N}$, $\eta(x) = 0, 1$ being the occupation number at $x \in \Lambda_N$. Note that

$$\epsilon := \frac{1}{N}.$$

and for the analysis in the following sections we use both N and ϵ to express quantities. Therefore, this slightly changes the notation we have used in the introduction. We shall study the Markov process on $\{0,1\}^{\Lambda_N}$ with generator

$$L_\epsilon := \epsilon^{-2}(L_0 + \epsilon L_{b_+} + \epsilon L_{b_-}) \tag{6.1}$$

where

$$L_0 f(\eta) := \frac{1}{2} \sum_{x=-N}^{N-1} [f(\eta^{x,x+1}) - f(\eta)] \tag{6.2}$$

$$L_{b_\pm} f(\eta) := \frac{j}{2} \sum_{x \in I_\pm} D_\pm \eta(x)[f(\eta^x) - f(\eta)], \tag{6.3}$$

η^x being the configuration obtained from η by changing the occupation number at x, $\eta^{x,x+1}$ by exchanging the occupation numbers at x, $x+1$; for any $u : \Lambda_N \to [0,1]$

$$D_+u(x) := [1-u(x)]u(x+1)u(x+2)\cdots u(N), \quad x \in I_+ \tag{6.4}$$

$$D_-u(x) := u(x)[1-u(x-1)][1-u(x-2)]\cdots[1-u(-N)], \quad x \in I_-. \tag{6.5}$$

L_0 is the generator of the SSEP (and of the stirring process as well) while $L_{b_\pm}$ are the generators of birth and deaths processes respectively. The former is active in I_+ and the latter in I_-.

We fix $T > 0$ and let $\mathcal{D}([0,T],\{0,1\}^{\Lambda_N})$ be the space of trajectories which are right continuous with left limits, and taking values in $\{0,1\}^{\Lambda_N}$. Denote by $\mathbb{P}^\epsilon_{\mu^\epsilon}$ the probability on $\mathcal{D}([0,T],\{0,1\}^{\Lambda_N})$ induced by the Markov process with generator L_ϵ and the initial measure μ^ϵ (it is a probability measure on $\{0,1\}^{\Lambda_N}$), and $\mathbb{E}^\epsilon_{\mu^\epsilon}$ the expectation with respect to $\mathbb{P}^\epsilon_{\mu^\epsilon}$. The expectation satisfies the following equation:

$$\frac{d}{dt}\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)] = \frac{1}{2}\Delta_\epsilon \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)] + \epsilon^{-1}\frac{j}{2}\left(\mathbb{1}_{x\in I_+}\mathbb{E}^\epsilon_{\mu^\epsilon}[D_+\eta_t(x)] - \mathbb{1}_{x\in I_-}\mathbb{E}^\epsilon_{\mu^\epsilon}[D_-\eta_t(x)]\right) \tag{6.6}$$

where $\Delta_\epsilon := \epsilon^{-2}\Delta$ the discrete Laplacian in Λ_N with reflecting boundary conditions:

$$\Delta u(x) = \nabla^+ u(x) - \nabla^- u(x), \quad x \neq \pm N \tag{6.7}$$

$$\Delta u(N) = -\nabla^- u(N) \tag{6.8}$$

$$\Delta u(-N) = \nabla^+ u(-N) \tag{6.9}$$

with

$$\nabla^+ u(x) = u(x+1) - u(x) \tag{6.10}$$

$$\nabla^- u(x) = u(x) - u(x-1) \tag{6.11}$$

Moreover, $\nabla^\pm_\epsilon := \epsilon^{-1}\nabla^\pm$.

6.2 Hydrodynamic Limit

We denote by $\rho_\epsilon(x,t)$, the solution of the following closed equation:

$$\frac{d}{dt}\rho_\epsilon(x,t) = \frac{1}{2}\Delta_\epsilon\rho_\epsilon(x,t) + \epsilon^{-1}\frac{j}{2}\left(\mathbb{1}_{x\in I_+}D_+\rho_\epsilon(x,t) - \mathbb{1}_{x\in I_-}D_-\rho_\epsilon(x,t)\right) \tag{6.12}$$

Theorem 6.1. *[22, Theorem 1] Suppose that the initial datum $\rho_\epsilon(\cdot,0)$ defined on Λ_N, with values in $[0,1]$ converges weakly as $\epsilon \to 0$ to $u_0(\cdot) \in L^\infty([-1,1],[0,1])$ in the sense*

$$\lim_{\epsilon\to 0} \epsilon \sum_{x\in\Lambda_N} \rho_\epsilon(\cdot,0)\phi(\epsilon x) = \int_{[-1,1]} u_0(r)\phi(r)dr, \quad \textit{for every } \phi \in L^\infty([-1,1],\mathbb{R}). \tag{6.13}$$

Then, there is $\rho(r,t)$, $r \in [-1,1]$, $t > 0$ so that for any $t_1 > t_0 > 0$

$$\lim_{\epsilon\to 0} \sup_{x\in\Lambda_N} \sup_{t_0\le t\le t_1} |\rho_\epsilon(x,t) - \rho(\epsilon x,t)| = 0 \tag{6.14}$$

The function $\rho(r,t)$ solves and is the unique solution of the integral equation

$$\begin{aligned} \rho(r,t) &= \int_{[-1,1]} P_t(r,r')u_0(r')dr' + \frac{j}{2}\int_{[-1,1]} P_s(r,1)(1-\rho(1,t-s)^K) \\ &\quad -P_s(r,-1)(1-(1-\rho(1,t-s))^K)ds \end{aligned} \tag{6.15}$$

where $P_t(r,r')$ is the density kernel of the semigroup (also denoted as P_t) with generator $\Delta/2$, Δ the Laplacian in $[-1,1]$ with reflecting, Neumann, boundary conditions.

Remark 6.2.

- The density kernel $P_t(r,r')$ can be expressed in terms of the Gaussian kernel

$$G_t(r,r') = \frac{e^{-(r-r')^2/2t}}{\sqrt{2\pi t}} \tag{6.16}$$

as follows: if $\psi : \mathbb{R} \to [-1,1]$ denotes the usual reflection map, i.e. $\psi(x) = x$ for $x \in [-1,1], \psi(x) = 2-x$ for $x \in [1,3]$, with ψ extended to the whole line as periodic of period 4, then

$$P_t(r,r') = \sum_{\psi(r'')=r'} G_t(r,r''), \quad \text{for } r,r' \neq \pm 1 \tag{6.17}$$

$$P_t(r,\pm 1) = \sum_{\psi(r'')=\pm 1} 2G_t(r,r'') \tag{6.18}$$

- Since ρ is smooth, we can write (6.15) in differential form: it then becomes the heat equation with Dirichlet boundary conditions:

$$\begin{cases} \partial_t\rho(r,t) = \frac{1}{2}\Delta\rho(r,t) & \text{for } t>0,\ r\in(-1,1), \\ \rho(-1,t) = u_-(t) & \text{for } t>0, \\ \rho(1,t) = u_+(t) & \text{for } t>0, \\ \rho(r,0) = u_0(r) & \text{for } r\in[-1,1]. \end{cases} \tag{6.19}$$

However, the boundary conditions $u_{\pm}(t)$ are not a priori known, they must be obtained by solving a non-linear system of two integral equations:

$$u_{\pm}(t) = \int_0^t \{p(s) f_{\pm}(u_{\pm}(t-s)) - q(s) f_{\mp}(u_{\mp}(t-s))\} \, ds + w_{\pm,t},$$

$$f_+(u) = \frac{j}{2}\left(1 - u^K\right), \quad f_-(u) = \frac{j}{2}\left(1 - (1-u)^K\right) \tag{6.20}$$

$$p(t) = \sum_{k\in\mathbb{Z}} G_t(4k), \quad q(t) = \sum_{k\in\mathbb{Z}} G_t(4k+2) \tag{6.21}$$

$$w_{+,t} = \sum_{k\in\mathbb{Z}} \int_{[-1,1]} u_0(r') G_t(1-r'+4k) dr', \quad w_{-,t} = \sum_{k\in\mathbb{Z}} \int_{[-1,1]} u_0(r') G_t(r'+1+4k) dr' \tag{6.22}$$

- By a simple computation one can check that

$$\left.\frac{\partial \rho(r,t)}{\partial r}\right|_{r=1} = j(1 - \rho(1,t)^K), \quad \left.\frac{\partial \rho(r,t)}{\partial r}\right|_{r=-1} = j(1 - (1-\rho(-1,t))^K) \tag{6.23}$$

6.3 Non-equilibrium fluctuations

6.3.1 Density Fluctuations

By $H \in C^\infty([-1,1])$ we mean that both $H : [-1,1] \to \mathbb{R}$, as well as all its derivatives are continuous functions in $[-1,1]$. Next, we consider a subspace of $C^\infty([-1,1])$ denoted by $\mathcal{C}$, which is intrinsically associated to the limiting fluctuations (see Remark 6.4) and we specify it in Sect. 6.4.1, Definition 6.12. Moreover, let $\mathcal{D}([0,T], \mathcal{C}')$ be the space of trajectories which are right continuous, with left limits and taking values in $\mathcal{C}'$. However, in order to define the fluctuation field formally, let us pretend that $\mathcal{C}$ is known for the moment and the reader could think of it as just a subspace of $C^\infty([-1,1])$.

Definition 6.3. We define the density fluctuation field $Y_\cdot^\epsilon(H)$ as the time-trajectory of linear functionals acting on functions $H \in \mathcal{C}$ as

$$Y_t^\epsilon(H) := \sqrt{\epsilon} \sum_{x=-N}^{N} H(\epsilon x) \left[\eta_t(x) - \mathbb{E}_{\mu^\epsilon}^\epsilon(\eta_t(x))\right] \tag{6.24}$$

For each $\epsilon > 0$, let $\mathbb{Q}_{\mu^\epsilon}^\epsilon$ be the probability measure on $\mathcal{D}([0,T], \mathcal{C}')$ induced by the density fluctuation field $Y_\cdot^\epsilon$ and the measure μ^ϵ.

Remark 6.4. As mentioned in the Introduction of this Part, there are terms in the explicit expression for (5.7) (see Sect. 6.4) that blow up. However, if $H \in \mathcal{C}$, they are eradicated and therefore, the choice of $\mathcal{C}$ together with the semigroup that the functions $H \in \mathcal{C}$ evolve and is denoted by T_t, (see Definition 6.73) seem the most suitable with respect to the computations we have done so far. This is explained in Sect. 6.4.1.

6.3.2 Discrete variance kernel of the density fluctuation field

In this section we compute the *discrete variance kernel* of the fluctuation field, $Y_t^\epsilon(H)$. Let us start by fixing some ideas.

Definition 6.5. We define

$$\bar{\eta}_t(x) := \eta_t(x) - \mathbb{E}^\epsilon_{\mu^\epsilon}(\eta_t(x))$$

and for every n positive integer,

$$C_t^{n,\epsilon}(x_1, \dots, x_n) := \mathbb{E}^\epsilon_{\mu^\epsilon}[\bar{\eta}_t(x_1) \dots \bar{\eta}_t(x_n)] \tag{6.25}$$

The next lemma is mainly based on the following:

$$\frac{\partial}{\partial t} C_t^{n,\epsilon}(x_1, \dots, x_n) = \mathbb{E}^\epsilon_{\mu^\epsilon}\left[L_\epsilon \prod_{i=1}^n \bar{\eta}_t(x_i)\right] + \mathbb{E}^\epsilon_{\mu^\epsilon}\left[\frac{\partial}{\partial t} \prod_{i=1}^n \bar{\eta}_t(x_i)\right]$$

for $n = 2$, where L_ϵ defined in (6.1).

Lemma 6.6. *The discrete kernel variance of the fluctuation field for the model is given by:*

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbb{E}^\epsilon_{\mu^\epsilon}\left[(Y_t^\epsilon(H))^2\right] \;=\;& T^{\neq}_{bulk,[|y-x|>1]} + T^{\neq}_{bulk,n.n} + T_{[|y-N|>1]} + T_{[|y+N|>1]} + T^{=}_{bulk} + T_N + \\
& + T_{-N} + T^{=}_{rbound} + T^{\neq}_{I_+^c, I_+} + T^{I_+}_{[x<y]} + T^{=}_{lbound} + T^{\neq}_{I_-^c, I_-} + T^{I_-}_{[x<y]} + \\
& + \epsilon\, j \sum_{x=-N}^{N} \sum_{y=N-K+1}^{N} H(\epsilon x) H(\epsilon y) \epsilon^{-1} R_\epsilon(x,y) + \\
& + \epsilon\, j \sum_{y=-N}^{N} \sum_{x=-N}^{-N+K-1} H(\epsilon x) H(\epsilon y) \epsilon^{-1} L_\epsilon(x,y)
\end{aligned} \tag{6.26}$$

where

$$T^{\neq}_{bulk,[y-x]>1} := \epsilon \sum_{x=-N+1}^{N-1} \sum_{|y-x|>1} H(\epsilon x) H(\epsilon y) \Delta_{\epsilon,x} C_t^{2,\epsilon}(x,y) \tag{6.27}$$

$$T^{\neq}_{bulk,n.n} := -\epsilon \sum_{x=-N}^{N-1} H(\epsilon x) H(\epsilon(x+1)) \left\{\nabla^+_\epsilon \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)]\right\}^2 \tag{6.28}$$

$$T_{[|y-N|>1]} := -\epsilon \sum_{|y-N|>1} H(1)H(\epsilon y)\nabla_\epsilon^- C_t^\epsilon(N,y) \tag{6.29}$$

$$T_{[|y+N|>1]} := \epsilon \sum_{|y+N|>1} H(-1)H(\epsilon y)\nabla_\epsilon^+ C_t^{2,\epsilon}(-N,y) \tag{6.30}$$

$$T_{bulk}^{=} := \frac{1}{2}\epsilon \sum_{x=-N+1}^{N-1} (H(\epsilon x))^2 \left(1 - 2\mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)]\right) \Delta_\epsilon \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)] \tag{6.31}$$

$$T_N := -\frac{1}{2}\epsilon\,(H(1))^2 \left(1 - 2\mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(N)]\right) \nabla_\epsilon^- \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(N)] \tag{6.32}$$

$$T_{-N} := \frac{1}{2}\epsilon\,(H(-1))^2 \left(1 - 2\mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(-N)]\right) \nabla_\epsilon^+ \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(-N)] \tag{6.33}$$

$$T_{rbound}^{=} := \frac{j}{2}\epsilon \sum_{x=N-K+1}^{N} (H(\epsilon x))^2 \left(1 - 2\mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)]\right) \epsilon^{-1} D_+ \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)] \tag{6.34}$$

$$T_{lbound}^{=} := -\frac{j}{2}\epsilon \sum_{x=-N}^{-N+K-1} (H(\epsilon x))^2 \left(1 - 2\mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)]\right) \epsilon^{-1} D_- \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)] \tag{6.35}$$

$$\begin{aligned} T_{I_+^c, I_+}^{\neq} := \epsilon\, j \sum_{x \notin [N-K+1,N]} \sum_{y=N-K+1}^{N} H(\epsilon x)H(\epsilon y)\epsilon^{-1}\Big(&- C_t^{2,\epsilon}(x,y) \prod_{z \in A(y)\setminus y} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z')] + \\ &+ \sum_{z \in A_+(y)} C_t^{2,\epsilon}(x,z) \prod_{z' \in A_+(y)\setminus z} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z')](1 - \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(y)]\Big) \end{aligned} \tag{6.36}$$

$$T_{[x<y]}^{I_+} := \epsilon\, j \sum_{\substack{x,y=N-K+1,\\ x<y}}^{N} H(\epsilon x)H(\epsilon y)\epsilon^{-1} \left(1 - \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(y)]\right) D_+ \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)] \tag{6.37}$$

$$\begin{aligned} T_{I_-^c, I_-}^{\neq} := \epsilon\, j \sum_{y \notin [-N,-N+K-1]} \sum_{x=-N}^{-N+K-1} H(\epsilon x)H(\epsilon y)\epsilon^{-1}\Big(& C_t^{2,\epsilon}(x,y) \prod_{z \in A_-(x)} (1 - \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z)]) + \\ & -\mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)] \sum_{z \in A_-(x)} C_t^{2,\epsilon}(y,z) \prod_{z' \in A_-(x)\setminus z} (1 - \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z')]\Big) \end{aligned} \tag{6.38}$$

$$T_{[x<y]}^{I_-} := -\mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x)] \sum_{z \in A_-(x)} C_t^{2,\epsilon}(y,z) \prod_{z' \in A_-(x)\setminus z} (1 - \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z')]) \tag{6.39}$$

Finally,

$$
\begin{aligned}
R_\epsilon(x,y) &:= \mathbb{1}_{x\in\Lambda_N, y\in I_+}\Big[\sum_{\substack{Z(y)\subseteq A_+(y),\\ |Z(y)|\geq 2}} C_t^{n_{Z(y)}+1,\epsilon}(Z(y),x) \prod_{z'\in A_+(y)\setminus Z(y)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')] - \\
&\quad - \sum_{\substack{B(y)\subseteq A(y),\\ |B(y)|\geq 2}} C_t^{n_{B(y)}+1,\epsilon}(B(y),x) \prod_{z'\in A(y)\setminus B(y)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')]\Big] + \\
&+ \quad \mathbb{1}_{x,y\in I_+}(1-\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)]) \\
&\quad \Big[\sum_{\substack{Z(x)\subseteq A_+(x),\\ |Z(x)|\geq 2}} C_t^{n_{Z(x)},\epsilon}(Z(x)) \prod_{z'\in A_+(x)\setminus Z(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')] - \\
&\quad - \sum_{\substack{B(x)\subseteq A(x),\\ |B(x)|\geq 2}} C_t^{n_{B(x)},\epsilon}(B(x)) \prod_{z'\in A(x)\setminus B(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')]\Big]
\end{aligned}
\tag{6.40}
$$

$$
\begin{aligned}
L_\epsilon(x,y) &:= \mathbb{1}_{y\in\Lambda_N, x\in I_-}\Big[\sum_{Z(x)\subseteq A_-(x)} (-1)^{|Z(x)|} \sum_{\substack{B(x)\subseteq Z(x),\\ B(x)\neq\emptyset}} \\
&\quad \prod_{z\in B(x)} C_t^{n_{B(x)}+2,\epsilon}(B(x),x,y) \prod_{z'\in Z(x)\setminus B(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')] + \\
&\quad + \sum_{Z(x)\subset A_-(x)} (-1)^{|Z(x)|} \sum_{\substack{B(x)\subseteq Z(x),\\ B(x)\neq\emptyset}} \\
&\quad \prod_{z\in B(x)} C_t^{n_{B(x)}+1,\epsilon}(B(x),y) \prod_{z'\in Z(x)\setminus B(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)]\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')] + \\
&- \quad \mathbb{1}_{x,y\in I_-}\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)]\Big(\sum_{Z(y)\subseteq A_-(y)} (-1)^{|Z(y)|} \sum_{\substack{B(y)\subseteq Z(y),\\ B(y)\neq\emptyset}} \\
&\quad \prod_{z\in B(y)} C_t^{n_{B(y)}+1,\epsilon}(B(y),y) \prod_{z'\in Z(y)\setminus B(y)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')] + \\
&\quad + D_-\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)] + \sum_{Z(y)\subset A_-(y)} (-1)^{|Z(y)|} \sum_{\substack{B(x)\subseteq Z(y),\\ B(y)\neq\emptyset}} \\
&\quad \prod_{z\in B(y)} C_t^{n_{B(y)},\epsilon}(B(y)) \prod_{z'\in Z(y)\setminus B(y)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)]\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z')]\Big)
\end{aligned}
\tag{6.41}
$$

where $A_+(y) = \{y+1,\dots,N\}$, $A(y) = \{y,\dots,N\}$ *and* $A_-(y) = \{-N,\dots,y-1\}$ *with* $n_{B(y)}$ *its cardinality.*

6.3.3 The v-estimates

In this subsection we present sharp bounds on the truncated correlation functions, the so called v-functions. We give the definition below:

Definition 6.7. Suppose that the process $\{\eta_t\}_{t\geq 0}$ starts with a product measure μ_ϵ and $\rho_\epsilon(x,t)$ is the solution of (6.12). We define as v-function

$$v_n^\epsilon(\underline{x},t|\mu^\epsilon) := \mathbb{E}_{\mu^\epsilon}^\epsilon\left[\prod_{i=1}^n(\eta_t(x_i)-\rho_\epsilon(x_i,t))\right], \quad \underline{x}\in\Lambda_N^{n,\neq}, n\geq 1 \tag{6.42}$$

where $\Lambda_N^{\neq}$ is the set of all sequences $(x_1,\ldots,x_n)\in\Lambda_N^n$ with $x_i\neq x_j$.

Theorem 6.8. *[23, Theorem 2.1] There exist $\tau>0$ and $c^*>0$ that the following holds. For any $\beta^*>0$ and for any positive integer n, there is a constant $c_n<\infty$ so that for every $\epsilon>0$, any initial product measure μ^ϵ,*

$$\sup_{\underline{x}\in\Lambda_N^{n,\neq}}|v_n^\epsilon(\underline{x},t|\mu^\epsilon)|\leq\begin{cases}c_n(\epsilon^{-2}t)^{-c^*n}, & \text{for } t\leq\epsilon^{\beta^*}\\ c_n\epsilon^{(2-\beta^*)c^*n}, & \text{for } \epsilon^{\beta^*}\leq t\leq\tau\log\epsilon^{-1}\end{cases} \tag{6.43}$$

As we would like to study the limit of (6.26) as $\epsilon\to 0$, we need to improve the estimate given in (6.43), and in particular we need an estimate as the next theorem states.

Theorem 6.9. *There exist $\tau>0$ that the following holds. For any positive integer n, there is a constant $c_n<\infty$ so that for every $\epsilon>0$, any initial product measure μ^ϵ,*

$$\sup_{\underline{x}\in\Lambda_N^{n,\neq}}|v_n^\epsilon(\underline{x},t|\mu^\epsilon)|\leq c_n\epsilon^{\frac{n}{2}}, \tag{6.44}$$

for every $t\leq\tau\log\epsilon^{-1}$.

To prove Theorem 6.9, one may follow the strategy in [25], *proof of Proposition 2.2.*

Remark 6.10. For every integer $n\geq 1$, $C_t^{n,\epsilon}$ is connected with the v-estimate in the following way:

$$\begin{aligned}\sup_{\underline{x}\in\Lambda_N^{n,\neq}}|C_t^{n,\epsilon}(\underline{x})| &\leq \sup_{\underline{x}\in\Lambda_N^{n,\neq}}|v_n^\epsilon(\underline{x},t|\mu^\epsilon)|+\binom{n}{n-1}\sup_{\underline{x}\in\Lambda_N^{n-1,\neq}}|v_{n-1}^\epsilon(\underline{x},t|\mu^\epsilon)|\sup_{\underline{x}\in\Lambda_N^{1,\neq}}|v_1^\epsilon(\underline{x},t|\mu^\epsilon)|\\ &+ \binom{n}{n-2}\sup_{\underline{x}\in\Lambda_N^{n-2,\neq}}|v_{n-2}^\epsilon(\underline{x},t|\mu^\epsilon)|\left(\sup_{\underline{x}\in\Lambda_N^{1,\neq}}|v_1^\epsilon(\underline{x},t|\mu^\epsilon)|\right)^2\\ &+ \cdots+\left(\sup_{\underline{x}\in\Lambda_N^{1,\neq}}|v_1^\epsilon(\underline{x},t|\mu^\epsilon)|\right)^n\end{aligned}$$

Hence, by using either Theorem 6.8 or Theorem 6.9, we get

$$\sup_{\underline{x}\in\Lambda_N^{n,\neq}}|C_t^{n,\epsilon}(\underline{x})|\leq\begin{cases}c_n(\epsilon^{-2}t)^{-c^*n}, & \text{for } t\leq\epsilon^{\beta^*}\\ c_n\epsilon^{(2-\beta^*)c^*n}, & \text{for } \epsilon^{\beta^*}\leq t\leq\tau\log\epsilon^{-1}\end{cases} \tag{6.45}$$

or

$$\sup_{\underline{x}\in\Lambda_N^{n,\neq}} |C_t^{n,\epsilon}(\underline{x})| \le c_n \epsilon^{\frac{n}{2}}, \quad \text{for every } t \le \tau \log \epsilon^{-1} \tag{6.46}$$

respectively, with c_n as in Theorem 6.8.

Once we have proven that the v-estimate given in (6.46) holds, then we can compute the limit of (6.26) as $\epsilon \to 0$. In the next subsection, we assume that such a v-estimate holds, and we give the explicit limiting equation for the discrete variance kernel for the fluctuation field, which leads to the *continuous variance kernel*.

6.3.4 Continuous Variance Kernel

We suppose that such a v-estimate given by (6.44) holds and consequently, the estimate (6.46) also holds. We also define the continuous variance kernel in the following way:

$$\lim_{\epsilon\to 0} \epsilon \sum_{x\in\Lambda_N} \sum_{y\in\Lambda_N} H(\epsilon x) H(\epsilon y) C_t^{2,\epsilon}(x,y) = \int_{-1}^{1}\int_{-1}^{1} H(r)H(u) C_t^*(r,u)\,dr\,du \tag{6.47}$$

Theorem 6.11. *The kernel variance of the limiting fluctuation field is given by:*

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} H(r)H(u)\frac{\partial}{\partial t} C_t^*(r,u)\,dr\,du &= \int_{-1}^{1}\int_{-1}^{1} H(r)H(u)\Delta_r C_t(r,u)\,dr\,du - \\
&- \int_{-1}^{1} (H(r))^2 (\partial_r \rho_t(r))^2 dr + \frac{1}{2}\int_{-1}^{1} (H(r))^2 (1-2\rho_t(r))\Delta_r \rho_t(r) dr + \\
&- \int_{-1}^{1} H(1)H(r)\partial_r C_t(1,r) dr + \int_{-1}^{1} H(-1)H(r)\partial_r C_t(-1,r) dr + \\
&- \frac{1}{2}(H(1))^2(1-2\rho_t(1))\partial_r\rho_t(1) + \frac{1}{2}(H(-1))^2(1-2\rho_t(-1))\partial_r\rho_t(-1) + \\
&+ \frac{j}{2}(H(1))^2(1-2\rho_t(1))(1-\rho_t(1)^K) - \\
&- \frac{j}{2}H(-1))^2(1-2\rho_t(-1))(1-(1-\rho_t(1)^K) + \\
&+ jK\int_{-1}^{1} H(1)H(r)C_t(1,r)\rho_t(1)^{K-1} dr + \\
&+ jK\int_{-1}^{1} H(-1)H(r)C_t(-1,r)(1-\rho_t(-1))^{K-1} dr \\
&+ jH(1)^2\rho_t(1)(1-\rho_t(1))(1-\rho_t(1)^{K-1}) \\
&+ jH(-1)^2\rho_t(-1)(1-\rho_t(-1))(1-(1-\rho_t(-1))^{K-1}).
\end{aligned} \tag{6.48}$$

Proof. Equation (6.48) is the limit of (6.26) as $\epsilon \to 0$. A way to see this is to approximate the integral by a Riemann sum, using also the v-estimate that as we have pointed out, it is assumed to be valid. □

6.4 Martingale Decomposition

We consider the following *martingale decomposition*, as in e.g. [33]. For $H : [-1,1] \times [0,T] \to \mathbb{R}$ a test function, we define

$$\mathcal{M}_t^\epsilon(H) := Y_t^\epsilon(H) - Y_0^\epsilon(H) - \int_0^t \Lambda_s^\epsilon(H)ds \tag{6.49}$$

$$\mathcal{N}_t^\epsilon(H) := (\mathcal{M}_t^\epsilon(H))^2 - \int_0^t \Gamma_s^\epsilon(H)ds, \tag{6.50}$$

where

$$\Lambda_s^\epsilon(H) := (\partial_s + L_\epsilon)Y_s^\epsilon(H) \tag{6.51}$$

$$\Gamma_s^\epsilon(H) := L_\epsilon(Y_s^\epsilon(H))^2 - 2Y_s^\epsilon(H)L_\epsilon Y_s^\epsilon(H). \tag{6.52}$$

These are martingales with respect to the natural filtration $\mathcal{F}_t := \sigma(\{\eta_s : s \leq t\})$. By computing explicitly the above martingales, we get

$$\begin{aligned}\Gamma_t^\epsilon(H) &= \epsilon \sum_{x=-N}^{N} (\nabla_\epsilon^+ H(\epsilon x)^2(\eta_t(x) - \eta_t(x+1))^2 + \\ &+ \frac{j}{2}\sum_{x\in I_+}(H(\epsilon x))^2 D_+\eta_t(x) + \frac{j}{2}\sum_{x\in I_-}(H(\epsilon x)^2 D_-\eta_t(x)\end{aligned} \tag{6.53}$$

and

$$\begin{aligned}\Lambda_t^\epsilon(H) &= Y_t^\epsilon(\partial_t H) + \sqrt{\epsilon}\sum_{x=-N+1}^{N-1}\frac{1}{2}\Delta_\epsilon H(\epsilon x)\bar{\eta}_t(x) - \\ &-\frac{1}{2\sqrt{\epsilon}}\nabla_\epsilon^- H(1)\bar{\eta}_t(N) + \frac{1}{2\sqrt{\epsilon}}\nabla_\epsilon^+ H(-1)\bar{\eta}_t(-N) + \\ &+\frac{1}{2\sqrt{\epsilon}}j\sum_{x=N-K+1}^{N-1} H(\epsilon x)\left(D_+\eta_t(x) - \mathbb{E}_{\mu^\epsilon}^\epsilon[D_+\eta_t(x)]\right) + \\ &-\frac{1}{2\sqrt{\epsilon}}j\sum_{x=-N+1}^{-N+K-1} H(\epsilon x)\left(D_-\eta_t(x) - \mathbb{E}_{\mu^\epsilon}^\epsilon[D_-\eta_t(x)]\right) - \\ &-\frac{1}{2\sqrt{\epsilon}}jH(1)\bar{\eta}_t(N) - \frac{1}{2\sqrt{\epsilon}}jH(-1)\bar{\eta}_t(-N)\end{aligned} \tag{6.54}$$

We analyse the sum of the fifth and the seventh term of (6.54) by following similar computations as in subsection 6.3.2.

$$\begin{aligned}\sum_{x=N-K+1}^{N-1} H(\epsilon x)D_+\eta(x) - H(1)\bar{\eta}_t(N) &= H(\epsilon(N-K+1))\prod_{z\in I_+}\eta_t(z) - \\ &- \sum_{x=N-K+1}^{N-1}\epsilon\nabla_\epsilon^- H(\epsilon x)\prod_{z\in A(x)}\eta_t(z),\end{aligned} \tag{6.55}$$

where $A(x) = \{x, \dots, N\}$. We analyse only the first term of the r.h.s. of (6.55) since the second term of r.h.s. of (6.55) multiplied by $\epsilon\nabla_\epsilon^-$ vanishes in the limit. We recall that $\rho_\epsilon(x,t)$ is the solution of (6.12), which for simplicity from now on we denote it by $\rho_t^\epsilon(x)$.

$$
\begin{aligned}
\prod_{z\in I_+} \eta_t(z) &= \sum_{Z(x)\subseteq I_+} \prod_{z\in Z(x)} \bar{\eta}_t(z) \prod_{y\in I_+\setminus Z(x)} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(y)] \\
&= \sum_{\substack{Z(x)\subseteq I_+,\\ |Z(x)|\neq 1}} \prod_{z\in Z(x)} \bar{\eta}_t(z) \prod_{y\in I_+\setminus Z(x)} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(y)] \\
&+ \sum_{x\in I_+} (\eta_t(x) - \eta_t^\epsilon(N)) \prod_{z\in I_+\setminus x} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z)] + \\
&+ \sum_{x\in I_+} \bar{\eta}_t(N) \sum_{\substack{Z(x)\subseteq I_+,\\ |Z(x)|\neq 0}} \prod_{z\in Z(x)} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z) - \eta_t(N)] \rho_t^\epsilon(N)^{K-1} + \\
&+ K\bar{\eta}_t(N)\rho_t^\epsilon(N)^{K-1}.
\end{aligned}
\tag{6.56}
$$

The expected value of (6.56) is given by

$$
\begin{aligned}
\mathbb{E}_{\mu^\epsilon}^\epsilon\left[\prod_{z\in I_+} \eta_t(z)\right] &= \sum_{\substack{Z(x)\subseteq I_+,\\ |Z(x)|\neq 1}} \prod_{z\in Z(x)} C_t^{|Z(x)|,\epsilon}(\underline{z}(Z)) \prod_{y\in I_+\setminus Z(x)} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(y)] + \\
&+ \sum_{x\in I_+} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x) - \eta_t^\epsilon(N)] \prod_{z\in I_+\setminus x} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z)],
\end{aligned}
\tag{6.57}
$$

where $C_t^{|Z(x)|,\epsilon}(Z(x))$ is defined in (6.25). By putting together (6.55), (6.56),(6.57), we get

$$
\begin{aligned}
\sum_{x=N-K+1}^{N-1} H(\epsilon x)\left(D_+\eta_t(x) - \mathbb{E}_{\mu^\epsilon}^\epsilon[D_+\eta_t(x)]\right) - H(1)\bar{\eta}_t(N) &= H(1)K\bar{\eta}_t(N)\rho_t^\epsilon(N)^{K-1} + \\
+ \; H(\epsilon(N-K+1))\Big(&\sum_{\substack{Z(x)\subseteq I_+,\\ |Z(x)|\neq 1}} \prod_{z\in Z(x)} \bar{\eta}_t(z) \prod_{y\in I_+\setminus Z(x)} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(y)] \\
+ &\sum_{x\in I_+} (\eta_t(x) - \eta_t^\epsilon(N)) \prod_{z\in I_+\setminus x} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z)] + \\
+ &\sum_{x\in I_+} \bar{\eta}_t(N) \sum_{\substack{Z(x)\subseteq I_+,\\ |Z(x)|\neq 0}} \prod_{z\in Z(x)} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z) - \eta_t(N)] \rho_t^\epsilon(N)^{K-1} + \\
+ &\sum_{\substack{Z(x)\subseteq I_+,\\ |Z(x)|\neq 1}} \prod_{z\in Z(x)} C_t^{|Z(x)|,\epsilon}(Z(x)) \prod_{y\in I_+\setminus Z(x)} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(y)] + \\
+ &\sum_{x\in I_+} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(x) - \eta_t^\epsilon(N)] \prod_{z\in I_+\setminus x} \mathbb{E}_{\mu^\epsilon}^\epsilon[\eta_t(z)]\Big) + \\
+ \; \epsilon\nabla_\epsilon^- H(1)K\bar{\eta}_t(N)\rho_t^\epsilon(N)^{K-1} &- \sum_{x=N-K+1}^{N-1} \epsilon\nabla_\epsilon^- H(\epsilon x)\mathbb{E}_{\mu^\epsilon}^\epsilon\left[\prod_{z\in A(x)} \eta_t(z)\right]
\end{aligned}
\tag{6.58}
$$

We have to keep in mind the first term of the r.h.s. of (6.58) multiplied by the factor provided in (6.54), that is

$$\frac{1}{2\sqrt{\epsilon}} H(1) K \bar{\eta}_t(N) \rho_t^\epsilon(N)^{K-1}$$

as it will play a crucial in the definition of the space of test functions $\mathcal{C}$. Next, we analyse the sum of the sixth and the eighth term of (6.54) in an analogous way.

$$\begin{aligned} D_-\eta_t(x) &= \sum_{Z(x)\subseteq A_-(x)} (-1)^{n^-(x,m_x)} \prod_{z\in Z(x)} \bar{\eta}_t(x)\bar{\eta}_t(z) \prod_{y\in A_-(x)\setminus Z(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)] \\ &+ \sum_{Z(x)\subseteq A_-(x)} (-1)^{n^-(x,m_x)} \prod_{z\in Z(x)} \bar{\eta}_t(z) \prod_{y\in A_-(x)\setminus Z(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)]\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)], \end{aligned} \tag{6.59}$$

where $A_-(x) := \{-N, \cdots, x-1\}$ and as before the first sum on each term is w.r.t to all subsets $Z(x)$ of $A_-(x)$. Moeover, for every $Z(x) \subseteq A_-(x)$, we define $n^-(x, m_x) = |A_-(x)| - |Z(x)|$ with $m_x = |Z(x)|$. Its expected value is also computed below

$$\begin{aligned} \mathbb{E}^\epsilon_{\mu^\epsilon}[D_-\eta_t(x)] &= \sum_{\substack{Z(x)\subseteq A_-(x),\\ Z(x)\neq\emptyset}} (-1)^{n^-(x,m_x)} C^{m_x+1,\epsilon}(x, \underline{z}(Z(x))) + \\ &+ \sum_{\substack{Z(x)\subseteq A_-(x),\\ |Z(x)|\neq 1}} (-1)^{n^-(x,m_x)} C^{m_x,\epsilon}(\underline{z}(Z(x))) \\ &\qquad \prod_{y\in A_-(x)\setminus Z(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)]\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)], \end{aligned} \tag{6.60}$$

and we recall that $\underline{z}(Z(x)) = \{z : z \in Z(x)\}$. Also note that the first term of (6.59), when $Z(x) = \emptyset$, can be written as

$$\begin{aligned} \bar{\eta}_t(x) \prod_{y\in A_-(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)] &= (\bar{\eta}_t(x) - \bar{\eta}_t(-N)) \prod_{z\in A_-(x)} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z)] + \bar{\eta}_t(-N) \sum_{\substack{Z(x)\subseteq A_-(x),\\ Z(x)\neq\emptyset}} \\ &\prod_{z\in Z(x)} \left(\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(z)] - \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(-N)]\right) \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(-N)]^{n^-(x,m_x)} + \\ &+ \bar{\eta}_t(-N)\rho_t^\epsilon(-N)^{n^-(x,0)} + \bar{\eta}_t(-N) v_1^\epsilon(-N, t|\mu^\epsilon) \end{aligned} \tag{6.61}$$

We recall that $n^-(x,0) = |A_-(x)|$ as in this case the cardinality of $Z(x)$ is 0. The second term of (6.59), when $|Z(x)| = 1$, can be written as

$$
\begin{aligned}
&\sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)}\bar{\eta}_t(\tilde{x}) \prod_{y\in A_-(x)\setminus\tilde{x}} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(x)]\mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)] = \\
&\quad = \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)}\bar{\eta}_t(\tilde{x}) v_1^\epsilon(x,t|\mu^\epsilon) \prod_{y\in A_-(x)\setminus\tilde{x}} \mathbb{E}^\epsilon_{\mu^\epsilon}[\eta_t(y)] + \\
&\quad + \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)}\bar{\eta}_t(\tilde{x})\rho_t^\epsilon(x) \sum_{\substack{Y(x)\subseteq A_-(x)\setminus\tilde{x},\\ Y(x)\neq\emptyset}} \prod_{y\in Y(x)} v_1^\epsilon(y,t|\mu^\epsilon) \prod_{z\in A_-(x)\setminus Y(x),y} \rho_t^\epsilon(z) + \\
&\quad + \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)}(\bar{\eta}_t(\tilde{x}) - \bar{\eta}_t(-N))\rho_t^\epsilon(x) \prod_{y\in A_-\setminus\tilde{x}} \rho_t^\epsilon(y) + \\
&\quad + \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)}\bar{\eta}_t(-N)\rho_t^\epsilon(x) \sum_{\substack{Y(x)\subseteq A_-(x)\setminus\tilde{x},\\ Y(x)\neq\emptyset}} \\
&\qquad \prod_{y\in Y(x),x} (\rho_t^\epsilon(y) - \rho_t^\epsilon(-N))\rho_t^\epsilon(-N)^{n^-(x,m_x)-2} \\
&\quad + \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)}\bar{\eta}_t(-N)\rho_t^\epsilon(-N)^{n^-(x,0)-2}
\end{aligned}
\tag{6.62}
$$

In addition,

$$
\begin{aligned}
&\sum_{x=-N+1}^{-N+K-1} (-1)^{n^-(x,0)} H(x)\bar{\eta}_t(-N)\rho_t^\epsilon(-N)^{n^-(x,0)} + H(-1)\bar{\eta}_t(-N) \\
&\qquad + \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)}\bar{\eta}_t(N)\rho_t^\epsilon(x)\rho_t^\epsilon(-N)^{n^-(x,0)-2} = \\
&\qquad\quad = H(-1)K(1-\rho_t^\epsilon(-N))^{K-1}\bar{\eta}_t(-N) + \\
&\qquad\quad + \bar{\eta}_t(-N)\sum_{x=-N+1}^{-N+K-1} \epsilon\nabla^-_\epsilon H(x)(\rho_t^\epsilon(-N)^{n^-(x,0)} + \rho_t^\epsilon(-N)^{n^-(x,0)-2})
\end{aligned}
\tag{6.63}
$$

By substituting (6.59), (6.60), (6.61) and (6.63) into the sum of the sixth and eighth terms of the r.h.s of (6.54), we get

$$
\begin{aligned}
&\sum_{x=-N+1}^{-N+K-1} H(\epsilon x)\left(D_-\eta_t(x) - \mathbb{E}^\epsilon_{\mu^\epsilon}[D_-\eta_t(x)]\right) + H(-1)\bar{\eta}_t(-N) = \\
&\qquad = H(-1)K(1-\rho_t^\epsilon(-N))^{K-1}\bar{\eta}_t(-N) + \\
&\qquad + \bar{\eta}_t(-N)\sum_{x=-N+1}^{-N+K-1} (-1)^{n^-(x,0)}\epsilon\nabla^-_\epsilon H(x)(\rho_t^\epsilon(-N)^{n^-(x,0)} + \rho_t^\epsilon(-N)^{n^-(x,0)-2}) + \\
&\qquad + \bar{\eta}_t(-N)v_1^\epsilon(-N,t|\mu^\epsilon)\sum_{x=-N+1}^{-N+K-1} (-1)^{n^-(x,0)} +
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{x=-N+1}^{-N+K-1} (-1)^{n^-(x,0)} H(\epsilon x)(\bar{\eta}_t(x) - \bar{\eta}_t(-N)) \prod_{z\in A_-(x)} \mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(z)] + \\
&+ \bar{\eta}_t(-N) \sum_{x=-N+1}^{-N+K-1} (-1)^{n^-(x,0)} H(\epsilon x) \sum_{\substack{Z(x)\subseteq A_-(x),\\ Z(x)\neq\emptyset}} \\
&\quad \prod_{z\in Z(x)} \left(\mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(z)] - \mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(-N)]\right) \mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(-N)]^{n^-(x,m_x)} + \\
&+ \sum_{x=-N+1}^{-N+K-1} \sum_{Z(x)\subseteq A_-(x)} (-1)^{n^-(x,m_x)} \prod_{z\in Z(x)} \bar{\eta}_t(x)\bar{\eta}_t(z) \prod_{y\in A_-(x)\setminus Z(x)} \mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(y)] \\
&+ \sum_{x=-N+1}^{-N+K-1} \sum_{Z(x)\subseteq A_-(x)} (-1)^{n^-(x,m_x)} \prod_{z\in Z(x)} \bar{\eta}_t(z) \\
&\quad \prod_{y\in A_-(x)\setminus Z(x)} \mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(x)]\mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(y)] + \\
&+ \sum_{x=-N+1}^{-N+K-1} \sum_{\substack{Z(x)\subseteq A_-(x),\\ Z(x)\neq\emptyset}} (-1)^{n^-(x,m_x)} C^{|Z(x)|+1,\epsilon}(x, \underline{z}(Z(x))) + \\
&+ \sum_{x=-N+1}^{-N+K-1} \sum_{\substack{Z(x)\subseteq A_-(x),\\ |Z(x)|\neq 0,1}} (-1)^{n^-(x,m_x)} C^{|Z(x)|,\epsilon}(\underline{z}(Z(x))) \\
&\quad \prod_{y\in A_-(x)\setminus Z(x)} \mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(x)]\mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(y)] + \\
&+ \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)} \bar{\eta}_t(\tilde{x}) v^{\epsilon}_1(x,t|\mu^\epsilon) \prod_{y\in A_-(x)\setminus\tilde{x}} \mathbb{E}^{\epsilon}_{\mu^\epsilon}[\eta_t(y)] + \\
&+ \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)} \bar{\eta}_t(\tilde{x}) \rho^{\epsilon}_t(x) \\
&\quad \sum_{\substack{Y(x)\subseteq A_-(x)\setminus\tilde{x},\\ Y(x)\neq\emptyset}} \prod_{y\in Y(x)} v^{\epsilon}_1(y,t|\mu^\epsilon) \prod_{z\in A_-(x)\setminus Y(x),y} \rho^{\epsilon}_t(z) + \\
&+ \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)} (\bar{\eta}_t(\tilde{x}) - \bar{\eta}_t(-N))\rho^{\epsilon}_t(x) \prod_{y\in A_-\setminus\tilde{x}} \rho^{\epsilon}_t(y) + \\
&+ \sum_{\tilde{x}\in A_-(x)} (-1)^{n^-(x,1)} \bar{\eta}_t(-N)\rho^{\epsilon}_t(x) \sum_{\substack{Y(x)\subseteq A_-(x)\setminus\tilde{x},\\ Y(x)\neq\emptyset}} \\
&\quad \prod_{y\in Y(x),x} (\rho^{\epsilon}_t(y) - \rho^{\epsilon}_t(-N))\rho^{\epsilon}_t(-N)^{n^-(x,m_x)-2}
\end{aligned}
\tag{6.64}
$$

In the above computation, the first term of the r.h.s is the one that we have to keep in mind, that is

$$\frac{1}{2\sqrt{\epsilon}} H(-1)K(1-\rho^{\epsilon}_t(-N))^{K-1}\bar{\eta}_t(-N)$$

As we mentioned in the end of Sect. 5.0.2, $\Lambda_t^\epsilon(H)$ consists of terms that diverge due to the factor $\frac{1}{2\sqrt{\epsilon}}$ in front of them, and terms which may naturally be vanishing in the limit. According to the formulas (6.54), (6.64) and (6.58), the divergent terms are the third [resp. forth] term of the r.h.s of (6.54) together with the first term of the r.h.s of (6.58) [resp. first term of (6.64)], that is

$$-\frac{1}{2\sqrt{\epsilon}}\left(\nabla_\epsilon^- H(1)) - H(1)\,Kj\,\rho_t^\epsilon(N)^{K-1}\right)\bar{\eta}_t(N) \tag{6.65}$$

$$\frac{1}{2\sqrt{\epsilon}}\left(\nabla_\epsilon^+ H(-1)) + H(-1)\,Kj\,(1-\rho_t^\epsilon(N))^{K-1}\right)\bar{\eta}_t(-N) \tag{6.66}$$

As we will see in the next section, by imposing $H \in \mathcal{C}$, these terms are eliminated in the limit. Apart from the forth term and the third term of the r.h.s of (6.58) and (6.64) respectively, that vanishes in the limit due to the gradient, the remaining terms in (6.58) and (6.64) should be vanishing using correlation estimates as the estimate (6.46) or even v-estimates at different times. Finally, the first and the second term of (6.54) is already expressed in a closed form.

6.4.1 Choice of Space of Test Functions

In this section we give an insight into the choice of space of test functions. Let us start by integrating by parts the first term of the r.h.s of (6.48):

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1} H(r)\,H(u)\Delta_r C_t(r,u)drdu &= \int_{-1}^{1}\int_{-1}^{1}\Delta_r H(r)H(u)C_t(r,u)drdu + \\ &+ \int_{-1}^{1} H(1)H(u)\partial_r C_t(1,u)du - \int_{-1}^{1} H(-1)H(u)\partial_r C_t(-1,u)du - \\ &- \int_{-1}^{1}\partial_r H(1)H(u)C_t(1,u)du + \int_{-1}^{1}\partial_r H(-1)H(u)C_t(-1,u)du \end{aligned} \tag{6.67}$$

The second and the third term of (6.67) are cancelled out with the forth and the fifth term of (6.48) respectively. It is easy to see that we can write the second and the third term of (6.48) the following way:

$$\begin{aligned}-\int_{-1}^{1}(H(r))^2(\partial_r\rho_t(r))^2dr &+ \frac{1}{2}\int_{-1}^{1}(H(r))^2(1-2\rho_t(r))\Delta_r\rho_t(r)dr \\ &= \frac{1}{2}\int_{-1}^{1}(H(r))^2\Delta_r(\chi(\rho_t(r)))dr\end{aligned}$$

Hence, we integrate by parts $\int_{-1}^{1}(H(r))^2\Delta_r(\chi(\rho_t(r)))dr$ and we get:

$$
\begin{aligned}
\frac{1}{2}\int_{-1}^{1}(H(r))^2\,\Delta_r(\chi(\rho_t(r)))dr = & \int_{-1}^{1}(\partial_r H(r))^2\chi(\rho_t(r))dr + \\
& + \int_{-1}^{1} H(r)\Delta_r H(r)\chi(\rho_t(r))dr + \frac{1}{2}(H(1))^2(1-2\rho_t(1))\partial_r\rho_t(1) - \\
& - \frac{1}{2}(H(-1))^2(1-2\rho_t(-1))\partial_r\rho_t(-1) - \\
& - H(1)\partial_r H(1)\chi(\rho_t(1)) + H(-1)\partial_r H(-1)\chi(\rho_t(-1)) \qquad (6.68)
\end{aligned}
$$

The sixth and the seventh term of (6.48) are cancelled out with the third and the forth term of (6.68) respectively. Moreover, the sum of the first term of (6.67) and the second term of (6.68) is equal to $\int_{-1}^{1}\int_{-1}^{1}\Delta_r H(r)H(u)C_t^*(r,u)drdu$. By taking into account all the above computations, (6.48) takes the form:

$$
\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} H(r)H(u)\frac{\partial}{\partial t}C_t^*(r,u)drdu = & \int_{-1}^{1}\int_{-1}^{1}\Delta_r H(r)H(u)C_t^*(r,u)drdu - \\
+ & \int_{-1}^{1}(\partial_r H(r))^2\chi(\rho_t(r))dr + \\
+ & \; j\int_{-1}^{1}\left(KH(1)\rho_t(1)^{K-1} - \partial_r H(1)\right)H(r)C_t(1,r)dr + \\
+ & \; j\int_{-1}^{1}\left(\partial_r H(-1) + KH(-1)(1-\rho_t(-1))^{K-1}\right)H(r)C_t(-1,r)dr \\
+ & \; \frac{j}{2}(H(1))^2(1-2\rho_t(1))(1-\rho_t(1)^K) + j(H(1))^2\rho_t(1)(1-\rho_t(1))(1-\rho_t(1)^{K-1}) - \\
- & \; H(1)\partial_r H(1)\chi(\rho_t(1)) - \\
- & \; \frac{j}{2}(H(-1))^2(1-2\rho_t(-1))(1-(1-\rho_t(-1))^K) + H(-1)\partial_r H(-1)\chi(\rho_t(-1)) \\
+ & \; j(H(-1))^2\rho_t(-1)(1-\rho_t(-1))(1-(1-\rho_t(-1))^{K-1}). \qquad (6.69)
\end{aligned}
$$

Let us look now at (6.53). It is easy to prove that $\Gamma_t^\epsilon(H)$ converges in law to

$$
\int_0^t\int_{-1}^{1}(\nabla H(r))^2\chi(\rho_t(r))drds + \frac{j}{2}\int_0^t H(1)^2\rho_t(1)^K ds + \frac{j}{2}\int_0^t H(-1)^2(1-(1-\rho_t(1))^K)ds \qquad (6.70)
$$

by using hydrodynamic limit, Theorem 6.1 and the correlation estimates provided by Theorem 6.9.

In [33], the authors prove for every $H \in C^\infty([-1,1])$, the sequence of martingales $\{\mathcal{M}_t^\epsilon(H) : t \in [0,T]\}_\epsilon$ converges (w.r.t to skorohod topology on $\mathcal{D}([0,T],\mathbb{R})$) to $\{\mathcal{W}_t(H) : t \in [0,T]\}$, which is Gaussian with mean zero and with a particular variance. On top of that, they prove that this variance under a suitable choice of test functions and a semigroup that the test functions evolve in and the extra assumption that the sequence of initial density fields $\{Y_0^\epsilon\}_\epsilon$, converges to a mean-zero Gaussian field with a specific covariance (see [33],Theorem 2.4), is indeed the variance for the limiting process. Following this idea, the expressions (6.69) and (6.48) should be the same. In other words, the analysis coming from direct computation of the variance at the microscopic scale and then passing to the limit (see Sect. 6.3.2 and Sect. 6.3.4), should conclude to the same result as the martingale analysis in Sect. 6.4. In fact, (6.70) and (6.69) can be equal if $H \in \mathcal{C}$ and T_t as in definitions below.

Definition 6.12. Let $\mathcal{C}$ denote the set of functions $H \in C^\infty([-1,1])$ that satisfy the following conditions at -1 and 1:

$$\partial_r H(1) = jK\rho_t(1)^{K-1}H(1), \tag{6.71}$$

$$\partial_r H(-1) = -jK(1-\rho_t(-1))^{K-1}H(-1), \tag{6.72}$$

where $K \geq 1$ and is the fixed integer we introduced in the beginning of the section and ρ_t is the solution of (6.15).

Remark 6.13. We have to impose some extra constrains in the definition regarding the higher derivatives at -1 and 1, provided by the proof of Theorem 5.13 for the model that we study here (see [33], definition 2.1).

Definition 6.14. Let $\mathcal{C}'$ the topological dual of $\mathcal{C}$ with respect to the topology generated by the seminorms

$$\|H\|_k = \sup_{r\in[-1,1]} |\partial_r^k H(r)|$$

where $k \in \mathbb{N} \cup 0$.

Definition 6.15. Let $T_t : \mathcal{C} \to \mathcal{C}$ be the associated semigroup to the following equation

$$\begin{cases} \partial_t u(r,t) = \Delta u(r,t) & \text{, for } t>0,\ r\in(-1,1), \\ \partial_r u(1,t) = jK\rho(1,t)^{K-1}u(1,t) & \text{, for } t>0, \\ \partial_r u(-1,t) = -jK(1-\rho(-1,t))^{K-1}u(-1,t) & \text{, for } t>0, \\ u(r,0) = u_\star(r) & \text{, for } r\in[-1,1]. \end{cases} \tag{6.73}$$

That is, given $u_\star \in \mathcal{C}$, by $T_t u_\star$ we mean the solution of (6.73) with initial condition $u_\star$.

To be specific, let $H : [0,T] \times [-1,1] \to \mathbb{R}$ be a test function, then it is easy to see that (6.69) has the extra term

$$\int_{-1}^{1}\int_{-1}^{1} \frac{\partial}{\partial t} H(r)H(u)C_t^*(r,u)drdu$$

In addition, if we choose

$$\phi(s,r) := T_{t-s}H(r) \tag{6.74}$$

for $H \in \mathcal{C}$, with T_t as in Definition 6.15, the variance of the limiting fluctuation field (6.69) has the form:

$$\begin{aligned} \int_{-1}^{1}\int_{-1}^{1} H(r)\,H(u)C_t^*(r,u)drdu = & \int_0^t \int_{-1}^{1} (\nabla T_{t-s}H(r))^2 \chi(\rho_t(r))drds + \\ + & \ \frac{j}{2}\int_0^t (T_{t-s}H(1))^2(\rho_t(1))^K ds + \\ + & \ \frac{j}{2}\int_0^t (T_{t-s}H(-1))^2(1-(1-\rho_t(1))^K)ds \end{aligned} \tag{6.75}$$

which is the same as (6.70) with the choice (6.74).

Summary/Remarks

1. In (5.0.2), we discussed that tightness is needed to characterise the limiting fluctuation field. In our model, to prove tightness we need to first prove Theorem 6.9.

2. To prove Theorem 6.9, one may follow the strategy in [25], *proof of Proposition 2.2*.

3. The choice (6.74) with $H \in \mathcal{C}$ make the terms (6.65) and (6.66) of $\Lambda_t^\epsilon(H)$ vanishing in the limit. For the remaining terms of $\Lambda_t^\epsilon(H)$, it is vague how to make them vanish, but we believe that space-time correlations may be needed for this. In particular, correlation estimates for different times and at the same site should also be studied. Once this is done, Theorem 5.13 can be applied, which also implies that the limiting process is the Ornstein-Uhlenbeck Process.

Bibliography

[1] D. Aldous, Stopping Times and Tightness, *Ann. Probab.*, **6** , no. 2, pp.335–340 (1978).

[2] R. Baldasso, O. Menezes, A. Neumann, R. Souza, Exclusion process with slow boundary, *J. Stat. Phys.* **167** , pp. 1112–1142 (2017).

[3] L. Bertini, P. Buttà, A. Pisante, Stochastic Allen-Cahn equation with mobility, *A. Nonlinear Differ. Equ. Appl.* **24**: 54 (2017).

[4] L. Bertini, P. Buttà, A. Pisante, Stochastic Allen-Cahn approximation of the mean curvature flow: large deviations upper bound, *A. Arch. Rational Mech. Anal.* **224** (2017), 659.

[5] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Macroscopic fluctuation theory, Rev. Mod. Phys. **87** (2015), 593.

[6] G. Bellettini, A. De Masi, E. Presutti, Energy levels of a non local functional, *J. Math. Phys.,* **46** (2005), no.8, 083302.

[7] G. Bellettini, A. De Masi, E. Presutti, Small, energy controlled perturbations of non local evolution equations, *preprint* (2004).

[8] G. Bellettini, A. De Masi, E. Presutti, Tunnelling for nonlocal evolution equations, *J. Nonlin. Math. Phys.,* **12**, 1, (2005), 50–63.

[9] L. Bertini, E. Presutti, B. Rüdiger, E. Saada, Dynamical fluctuations at the critical point: convergence to a nonlinear stochastic PDE, *Teor. Veroyatnost. i Primenen.* **38** (1993) 689–741, translation in *Theory Probab. Appl.* **38** (1993) 586–629.

[10] T. Bodineau, B. Derrida, Current large deviations for Asymmetric Exclusion Processes with open boundaries, *J. Stat. Phys.*,**123**, No. 2, 277–300 (2006)

[11] T. Bodineau, B. Derrida, J. L. Lebowitz, A diffusive system driven by a battery or by a smoothly varying field, *J. Stat. Phys.*,**140**, 648–675 (2010)

[12] P. Birmpa, N. Dirr, D. Tsagkarogiannis, Large deviations for the macroscopic motion of an interface, *J. Stat. Phys.* **166**, 1163–1192 (2017).

[13] P. Birmpa, P. Gonçalves, Tsagkarogiannis, Nonequilibrium fluctuations for current reservoirs, in progress (2018).

[14] P. Birmpa, D. Tsagkarogiannis, Action minimization and macroscopic interface motion under forced displacement, to appear in *ESAIM: COCV* (2017).

[15] A. Bovier, *Statistical mechanics of disordered systems*, Cambridge Series in Statistical and ProbabilisticMathematics. CambridgeUniversity Press, Cambridge, 2006. A mathematical perspective.

[16] A. Bovier, *Lecture notes Gibbs measures and phase transitions. Part 1*, Rheinische Friedrich-Wilhelms Universität Bonn Institut für Angewandte Mathematik.

[17] S. Brush, For the history of Ising model, *Reviews of Modern Physics*, **39** (1969), 883.

[18] F. Comets, Nucleation for a long range magnetic model, *Ann. Inst. H. Poincaré - Probab. Statist.* **23** (1987), 135–178.

[19] F. Comets and Th. Eisele, Asymptotic dynamics, noncritical and critical fluctuations for a geometric long-range interacting model, *Comm. Math. Phys.* **118** (1988), no. 4, 531–567.

[20] A. De Masi, N. Dirr, E. Presutti, Interface instability under forced displacements. *Ann. I. H. Poincaré - AN* **7** (2006), no. 3, 471–511.

[21] A. De Masi, E.Presutti, *Mathematical Methods for Hydrodynamic Limits*, Lecture Notes in Mathematics, vol. 1501. Springer, Berlin (1991).

[22] A. De Masi, E. Presutti, D. Tsagkarogiannis, M. Vares: Current Reservoirs in the Simple Exclusion Process. *J. Stat. Phys.* **144(6)** (2011), pp. 1151–1170.

[23] A. De Masi, E. Presutti, D. Tsagkarogiannis, and M. Vares: Truncated correlations in the stirring process with births and deaths. *Electron. J. Probab.* **17** (2012), no. 6, pp. 1–35.

[24] A. De Masi, E. Presutti, D. Tsagkarogiannis, M. Vares: Non-equilibrium Stationary States in the Symmetric Simple Exclusion with Births and Deaths. *J. Stat. Phys.* **147(3)** (2012), pp. 519–528.

[25] A. De Masi, E. Presutti, E. Scacciatelli, The weakly asymmetric simple exclusion process, *Annales de l'I.H.P. Probabilités et statistiques* **25**.1 (1989): 1–38.

[26] A. De Masi, E. Olivieri, E. Presutti, Spectral properties of integral operators in problems of interface dynamics and metastability, *Markov Process. Related Fields* **4** (1998), 27–112.

[27] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, Glauber evolution with the Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics, *Nonlinearity* **7** (1994), no. 3, 633–696.

[28] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, Stability of the interface in a model of phase separation, *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994), 1013–1022.

[29] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, Uniqueness and global stability of the instanton in non local evolution equations, *Rend. Mat. Appl. (7)* **14** (1994), 693–723.

[30] B. Derrida, J. L. Lebowitz, E. R. Speer, Large deviation fo the density profile in the steady state of the simple exclusion process, *J. Stat. Phys.*, **107**, 599–634 (2002)

[31] W. E, W. Ren, E. Vanden-Eijnden, Minimum action method for the study of rare events, *Comm. Pure. Appl. Math.* (2004), Vol. LVII, 0001–0020.

[32] W. Faris and G. Jona-Lasinio, Large fluctuations for a nonlinear heat equation with noise, *J. Phys. A: Math. Gen.* **15** (1982), 3025–3055.

[33] T. Franco, P. Goncalves, A. Neumann: Stationary and Nonequilibrium Fluctuations in Boundary Driven Exclusion Processes. *https://arxiv.org/abs/1608.04317*, (2017).

[34] T. Franco, P. Gonçalves, A. Neumann, Equilibrium Fluctuations for the Slow Boundary Exclusion Process. *From Particle Systems to Partial Differential Equations. PSPDE 2015. Springer Proceedings in Mathematics and Statistics, vol 209. Springer, Cham* (2017), pp 177–197.

[35] P. Ferrari, E. Presutti, M. E. Vares, *Non equilibrium fluctuations for a zero range process*, Annales de l'I.H.P. Probabilités et statistiques **24**.2 (1988): 237–268.

[36] S. Friedli, Y. Velenik, *Statistical Mechanics of Lattice Systems:A Concrete Mathematical Introduction*, Cambridge University Press (2017).

[37] M. I. Freidlin, A.D. Wentzell: *Random Perturbations of Dynamical Systems*. Springer-Verlag Vol. 260 (1984).

[38] G. Gallavotti, *Statistical Mechanics, a Short Treatise*, Text and Monographs in Physics, Springer, Berlin (1999).

[39] A. Galves, C. Kipnis, C. Marchioro, E. Presutti Nonequilibrium measures which exhibit a temperature gradient: Study of a model, *Commun. Math. Phys.* **81**, 127–147 (1981)

[40] H. O. Georgii, *Gibbs Measures and Phase Transitions*, W. de Gruyter (1988).

[41] S. Grosskinsky, Large-Scale Dynamics of Stochastic Particle Systems, Lecture notes, *go.warwick.ac.uk/SGrosskinsky/teaching/tcc.html*, Warwick (2017).

[42] P. Gonçalves, M. Jara, Stochastic Burgers equation from long range exclusion interactions, *Stochastic Processes and their Applications*, **127 (12)** (2014), pp 4029–4052.

[43] M. Hairer, H. Weber, Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions, *Ann. Fac. Sci. Toulouse Math. (6)* **24** (2015), no. 1, 55–92.

[44] P. C. Hohenberg and B. I. Halperin, Theory of dynamic critical phenomena. *Rev. Mod. Phys.* **49** (1977), 435–479.

[45] R. A. Holley, D. W. Stroock, *Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions*, Publ. RIMS. Kyoto Univ.**14** (1978), 741–788 .

[46] M. Jara, C. Landim, Nonequilibrium central limit theorem for a tagged particle in symmetric simple exclusion, *Ann. Inst. Henri Poincaré Prob. et Statistiques*, **42** (2006), pp 567–577.

[47] G. Jona-Lasinio, P.K. Mitter, Large deviations estimates in the stochastic quantization of ϕ_2^4, *Comm. Math. Phys.* **130** (1990), 111–121.

[48] M. Katsoulakis, P. Souganidis, Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interactions and Glauber dynamics, *Comm. Math. Phys.* **169** (1995), no. 1, 61–97.

[49] C. Kipnis, C. Landim, *Scaling limits of interacting particle systems*, Springer-Verlag Vol. 320 (1999).

[50] T. Komorowski, C.Landim, S. Olla, *Fluctuations in Markov Processes: Time Symmetry and Martingale Approximation*, Springer (2012).

[51] T. Kuna, G. Manzi, Tunneling time for an Ising spin system with Glauber dynamics, *MIS MPG preprint* no. 93 (2006).

[52] R. V. Kohn, F. Otto, M.G. Reznikoff, E. Vanden-Eijnden, Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation, *Comm. Pure Appl. Math.* **60** (2007), 393–438.

[53] R. V. Kohn, M. G. Reznikoff, Y. Tonegawa, Sharp-interface limit of the Allen-Cahn action functional in one space dimension, *Calc. Var. PDE* **25** (2006), 503–534.

[54] T. M. Liggett, *Interacting particle systems*, Classics in Mathematics, Springer-Verlag, Berlin, (2005), Reprint of the 1985 original.

[55] C. Landim, A. Milanés, S. Olla, Stationary and Nonequilibrium Fluctuations in Boundary Driven Exclusion Processes. *Markov Processes and Related Fields*, **14** (2008), pp. 165–184.

[56] C. Landim, M. Mourragui, S. Sellami, Hydrodynamic limit of nongradient systems in contact with stochastic reservoirs, *Teoriya Veroyatnostei i ee Primeneniya* **45**, 694–717, (2000) and Theory of Probability and Its Applications. **45**, 604–623, (2001).

[57] I . Mitoma. Tightness of Probabilities On $C([0,1];Y')$ and $\mathcal{D}([0,1];Y')$, *Ann. Probab.*, **11(4)**: 989–999, (1983).

[58] L. Mugnai, M. Röger, The Allen-Cahn action functional in higher dimensions, *Interfaces Free Bound.* **10** (2008), 45–78.

[59] J.-C. Mourrat, H. Weber, Convergence of the two-dimensional dynamic Ising-Kac model to Φ^4_2, *Comm. Pure Appl. Math.* online first (2014).

[60] E. Presutti, *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*, Springer (2000).

[61] F. Rezakhanlou, Hydrodynamic Limit for Attractive Particle Systems on $\mathbb{Z}^d$, *Commun. Math. Physics*, **140** (1991), 417–448.

[62] K. Ravishankar, Fluctuations from the hydrodynamical limit for the symmetric simple exclusion in $\mathbb{Z}^d$, *Stoc. Proc. Appl.*, **42** (1992), pp 31–37.

[63] K. Ravishankar, Interface fluctuations in the two-dimensional weakly asymmetric simple exclusion process, *Stoch. Proc. Appl*, **43** (1992b), 223–247.

[64] H. Spohn, Long range correlations for stochastic lattice gases in a non-equilibrium steady state, *J. Phys. A:Math. Gen.*, **16** (1983), 4275–4291.